Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 12 Issue 1(2021), Pages 14-22.

ON A DIFFERENCE CONCERNING THE NUMBER e AND SUMMATION IDENTITIES OF PERMUTATIONS

MEHDI HASSANI

Abstract. In this article we obtain the generating function of the sequence $P(n,j)$, the number of j-permutations of n objects. We consider two applications. Our method, which employs integration by parts, gives an integral representation for the difference $(\mathrm{e}-\mathrm{e}_n)\,n!$, where $\mathrm{e}_n=\sum_{j=0}^{n}\frac{1}{j!}$ denote the truncated expansion of the number e. By using this integral representation we obtain a sharp double sided inequality concerning the difference $(\mathrm{e}-\mathrm{e}_n)\,n!$, and also we compute some moments of it. As the second application, we provide a recurrence method to compute power weighted sums concerning $P(n,j)$, and also alternating power weighted sums of $P(n,j)$ in terms of the number of derangements.

1. Introduction and summary of the results

1.1. A factorial-weighted difference concerning the number e.

In this paper we study the sequence $(\delta_n)_{n\geqslant 1}$ defined by

$$\delta_n := (\mathrm{e}-\mathrm{e}_n)\,n!,$$

where e denote the natural logarithm base and

$$\mathrm{e}_n=\sum_{j=0}^{n}\frac{1}{j!}. \tag{1.1}$$

We observe that

$$0<\delta_n=n!\sum_{j=1}^{\infty}\frac{1}{(n+j)!}<\sum_{j=1}^{\infty}\frac{1}{(n+1)^j}=\frac{1}{n}\leqslant 1.$$

Thus, the double sided inequality

$$0<\delta_n<1 \tag{1.2}$$

holds for each $n\geqslant 1$. This inequality is the key of the proof of irrationality of the number e (see Page 65 of [9]). Motivated by obtaining sharper inequalities for this difference, in 2012 Mortici [7] proved that

$$\frac{x_n}{n}<\delta_n<\frac{y_n}{n} \tag{1.3}$$

2000 *Mathematics Subject Classification.* 05A05, 05A16, 26A06.
Key words and phrases. permutation; derangement; integration by parts.

Submitted June 30, 2020. Published January 17, 2021.
Communicated by V. Krasniqi.

for any integer $n \geqslant 1$, where x_n and y_n are two explicitly determined sequences, both tending to 1, at the rate $1 + O(n^{-2})$. In 2015 the author and Sofo [5] proved that

$$\frac{1}{n+\alpha} \leqslant \delta_n < \frac{1}{n+\beta}$$

for any integer $n \geqslant 1$, with the best possible constants $\alpha = \frac{1}{\mathrm{e}-2} - 1 \approxeq 0.39$ and $\beta = 0$. In this paper we prove the following result providing sharp inequalities for the difference under study.

Theorem 1.1. *Given any positive integer r, there exist computable constants $c_1, \ldots, c_r$ such that for any integer $n \geqslant 1$ we have*

$$\sum_{k=1}^{r} \frac{c_k}{n^k} - \frac{\mathrm{e}^2 B_{r+1}}{n^{r+1}} < \delta_n < \sum_{k=1}^{r} \frac{c_k}{n^k} + \frac{\mathrm{e}^2 B_{r+1}}{n^{r+1}}, \tag{1.4}$$

with B_k denoting the k-th Bell number.

Remark. *Dobinski's formula concerning the k-th Bell number asserts that (see Page 178 of* [12]*)*

$$\mathrm{e}\, B_k = \sum_{j=0}^{\infty} \frac{j^k}{j!}. \tag{1.5}$$

The constants c_k match the k-th term of the sequence A014182 on OEIS [11]*, and satisfy the following alternating form of Dobinski's formula*

$$\frac{(-1)^{k-1}}{\mathrm{e}}\, c_k = \sum_{j=0}^{\infty} (-1)^{j-1}\, \frac{j^k}{j!}.$$

Some initial values of c_k is $1, 0, -1, 1, 2, -9, 9, 50, -267, 413, \ldots$.

Since $c_2 = 0$, the above result implies the double sided inequality (1.3). More precisely, since $B_3 = 5$ we obtain.

Corollary 1.2. *The inequality* (1.3) *holds for any integer $n \geqslant 1$ with*

$$x_n = 1 - \frac{5\,\mathrm{e}^2}{n^2}, \qquad \textit{and} \qquad y_n = 1 + \frac{5\,\mathrm{e}^2}{n^2}.$$

To prove Theorem 1.1 we consider an integral relation between δ_n and the sum $\sum_{j=0}^{n} P(n,j)$, where as usual $P(n,j)$ denote the number of j-permutations of n objects, counting the number of ways to choose an ordered selection of j items from a set of n items. More precisely, we obtain generating function of the sequence $P(n,j)$ in the following result.

Theorem 1.3. *Let a be fixed real number, and for any integer $n \geqslant 0$ let*

$$E_n(a) = \int_{-\infty}^{a} t^n \mathrm{e}^t \,\mathrm{d}t.$$

Then, for any integer $n \geqslant 0$ and for each real $x \neq 0$ we have

$$S_n(x) := \sum_{j=0}^{n} P(n,j)\, x^j = (-1)^n\, x^n\, \mathrm{e}^{\frac{1}{x}}\, E_n\left(-\frac{1}{x}\right). \tag{1.6}$$

The relation (1.6) enables us to obtain an integral representation for δ_n. This integral representation is the key to prove (1.4), and also the key to study the moments of δ_n.

Theorem 1.4. *For each integer $k \geqslant 2$ the following multiple integral representation holds*

$$\sum_{n=0}^{\infty} \delta_n^k = \mathrm{e}^k \int_0^1 \cdots \int_0^1 \frac{\mathrm{e}^{-(x_1+\cdots+x_k)}}{1 - x_1 \cdots x_k} \, \mathrm{d}\mathbf{X}, \tag{1.7}$$

where $\mathbf{X}$ *represents the* k*-tuple* $(x_1, \ldots, x_k)$*. More precisely, for* $k = 2$ *we have*

$$\sum_{n=0}^{\infty} \delta_n^2 = 4\mathrm{e}^2 \int_0^{\frac{1}{2}} h(z) \, \mathrm{d}z \approxeq 4.0275, \tag{1.8}$$

where

$$h(z) = \frac{\mathrm{e}^{-2z}}{\sqrt{1-z^2}} \arctan \frac{z}{\sqrt{1-z^2}} + \frac{\mathrm{e}^{2z-2}}{\sqrt{2z-z^2}} \arctan \frac{z}{\sqrt{2z-z^2}}.$$

1.2. **Summation identities of the number of permutations.** Although there is a remarkable number of summation identities concerning $C(n, j)$, the number of j-combinations of n objects (for example see Section 0.15 of [2], Section 2.3.4 of [8], and pages 343–355 of [10] for a list of 334 identities), we find less summation identities concerning $P(n, j)$. The relation (1.6) enables us to obtain a family of summation identities concerning $P(n, j)$.

Theorem 1.5. *Given any positive integer* k*, there exist computable polynomials* $A_k(x)$ *and* $B_k(x)$ *such that for each* $n \geqslant 1$ *and for* $x \neq 0$ *we have*

$$\sum_{j=0}^{n} j^k \, P(n, j) \, x^j = \frac{A_k(x)}{x^k} + \frac{B_k(x)}{x^k} \, S_n(x), \tag{1.9}$$

where $S_n(x)$ *is defined in* (1.6) *and the polynomials* $A_k(x)$ *and* $B_k(x)$ *have the following properties:*

- *The coefficients of* $A_k(x)$ *and* $B_k(x)$ *are at most in terms of* n*,*
- $\deg A_k(x) = k - 1$ *and* $\deg B_k(x) = k$*,*
- $A_1(x) = 1$*,* $B_1(x) = nx - 1$
- *for* $k \geqslant 1$ *the polynomials* $A_k(x)$ *and* $B_k(x)$ *satisfy the simultaneous recurrence*

$$\begin{cases} A_{k+1}(x) = B_k(x) - kxA_k(x) + x^2 \dfrac{\mathrm{d}}{\mathrm{d}x} A_k(x), \\ \\ B_{k+1}(x) = (nx - kx - 1) \, B_k(x) + x^2 \dfrac{\mathrm{d}}{\mathrm{d}x} B_k(x). \end{cases} \tag{1.10}$$

By using the simultaneous recurrence (1.10) with initial values $A_1(x) = 1$ and $B_1(x) = nx - 1$ we may compute the polynomials $A_k(x)$ and $B_k(x)$ for each k, some of which as follows

$$A_2(x) = (n-1)\, x - 1,$$
$$B_2(x) = n^2 x^2 + (-2n+1)\, x + 1,$$

$$A_3(x) = \left(n^2 - n + 1\right) x^2 + (-2n+3)\, x + 1,$$
$$B_3(x) = n^3 x^3 + \left(-3n^2 + 3n - 1\right) x^2 + (3n-3)\, x - 1,$$

$$A_4(x) = \left(n^3 - n^2 + n - 1\right) x^3 + \left(-3n^2 + 7n - 7\right) x^2 + (3n-6)\, x - 1,$$
$$B_4(x) = n^4 x^4 + \left(-4n^3 + 6n^2 - 4n + 1\right) x^3 + \left(6n^2 - 12n + 7\right) x^2 + (-4n+6)\, x + 1.$$

For an analogue to the identity $\sum_{j=0}^{n} C(n,j) = 2^n$ concerning $P(n,j)$, we note that the inequality (1.2) implies

$$S_n(1) = \sum_{j=0}^{n} P(n,j) = n!\,\mathrm{e}_n = [\mathrm{e}\,n!]\,, \tag{1.11}$$

for each $n \geqslant 1$, where $[x]$ denotes the largest integer not exceeding x. This relation has combinatorial meaning and some applications concerning complete simple graphs (see [3]). By using the relation (1.9) with $x = 1$ we get

$$\sum_{j=0}^{n} j^k\, P(n,j) = A_k(1) + B_k(1)\, S_n(1).$$

Thus, by considering the relation (1.11), we obtain the following.

Corollary 1.6. *For each integer $n \geqslant 1$ we have*

$$\sum_{j=0}^{n} j^k\, P(n,j) = A_k(1) + B_k(1)\ [\mathrm{e}\,n!]\,,$$

where for $k = 1, \ldots, 8$ the values of $A_k(1)$ and $B_k(1)$, as some polynomials of n, are given in the following tables.

k	$A_k(1)$
1	1
2	$n-2$
3	$n^2 - 3\,n + 5$
4	$n^3 - 4\,n^2 + 11\,n - 15$
5	$n^4 - 5\,n^3 + 19\,n^2 - 44\,n + 52$
6	$n^5 - 6\,n^4 + 29\,n^3 - 93\,n^2 + 191\,n - 203$
7	$n^6 - 7\,n^5 + 41\,n^4 - 167\,n^3 + 478\,n^2 - 898\,n + 877$
8	$n^7 - 8\,n^6 + 55\,n^5 - 271\,n^4 + 988\,n^3 - 2601\,n^2 + 4547\,n - 4140$

k	$B_k(1)$
1	$n-1$
2	$n^2 - 2\,n + 2$
3	$n^3 - 3\,n^2 + 6\,n - 5$
4	$n^4 - 4\,n^3 + 12\,n^2 - 20\,n + 15$
5	$n^5 - 5\,n^4 + 20\,n^3 - 50\,n^2 + 75\,n - 52$
6	$n^6 - 6\,n^5 + 30\,n^4 - 100\,n^3 + 225\,n^2 - 312\,n + 203$
7	$n^7 - 7\,n^6 + 42\,n^5 - 175\,n^4 + 525\,n^3 - 1092\,n^2 + 1421\,n - 877$
8	$n^8 - 8\,n^7 + 56\,n^6 - 280\,n^5 + 1050\,n^4 - 2912\,n^3 + 5684\,n^2 - 7016\,n + 4140$

An analogue to the identity $\sum_{j=0}^{n} (-1)^j\, C(n,j) = 0$ concerning $P(n,j)$ is related to the number of derangements (a permutation that leaves no element fixed) on a set of cardinality n, which we denote by D_n. Indeed, for any integer $n \geqslant 1$ we have

$$D_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!} = (-1)^n\, n! \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!} = (-1)^n \sum_{j=0}^{n} (-1)^j\, P(n,j).$$

Thus,

$$S_n(-1) = \sum_{j=0}^{n} (-1)^j\, P(n,j) = (-1)^n\, D_n. \tag{1.12}$$

Among several known closed form formulas for the number of derangements (see [4]) we recall the following one, which is similar to the value of $S_n(1)$,

$$D_n = \left[\frac{n!+1}{\mathrm{e}}\right]. \tag{1.13}$$

By using the relation (1.9) with $x = -1$ we get

$$\sum_{j=0}^{n} (-1)^j \, j^k \, P(n,j) = A_k(-1) + B_k(-1)\, S_n(-1).$$

Thus, by considering the relations (1.12) and (1.13), we obtain the following.

Corollary 1.7. *For each integer $n \geqslant 1$ we have*

$$\sum_{j=0}^{n} (-1)^j \, j^k \, P(n,j) = A_k(-1) + (-1)^n \, B_k(-1) \left[\frac{n!+1}{\mathrm{e}}\right],$$

where for $k = 1, \dots, 8$ the values of $A_k(-1)$ and $B_k(-1)$, as some polynomials of n, are given in the following tables.

k	$A_k(-1)$
1	1
2	$-n$
3	$n^2 + n - 1$
4	$-n^3 - 2\,n^2 + 3\,n - 1$
5	$n^4 + 3\,n^3 - 5\,n^2 + 2$
6	$-n^5 - 4\,n^4 + 7\,n^3 + 5\,n^2 - 17\,n + 9$
7	$n^6 + 5\,n^5 - 9\,n^4 - 15\,n^3 + 52\,n^2 - 42\,n + 9$
8	$-n^7 - 6\,n^6 + 11\,n^5 + 31\,n^4 - 114\,n^3 + 95\,n^2 + 33\,n - 50$

k	$B_k(-1)$
1	$-n - 1$
2	$n^2 + 2\,n$
3	$-n^3 - 3\,n^2 + 1$
4	$n^4 + 4\,n^3 - 4\,n + 1$
5	$-n^5 - 5\,n^4 + 10\,n^2 - 5\,n - 2$
6	$n^6 + 6\,n^5 - 20\,n^3 + 15\,n^2 + 12\,n - 9$
7	$-n^7 - 7\,n^6 + 35\,n^4 - 35\,n^3 - 42\,n^2 + 63\,n - 9$
8	$n^8 + 8\,n^7 - 56\,n^5 + 70\,n^4 + 112\,n^3 - 252\,n^2 + 72\,n + 50$

Finally, we mention that the truth of Theorem 1.5 has a connection with the incomplete gamma function $\Gamma(\alpha, z)$, which is defined by

$$\Gamma(\alpha, z) = \int_z^{\infty} t^{\alpha-1} \mathrm{e}^{-t} \, \mathrm{d}t.$$

Note that

$$E_n(a) = (-1)^n \int_{-a}^{\infty} t^n \mathrm{e}^{-t} \, \mathrm{d}t = (-1)^n \, \Gamma(n+1, -a).$$

Thus, for any integer $n \geqslant 0$ and for each real $x \neq 0$ we have

$$S_n(x) = x^n \, \mathrm{e}^{\frac{1}{x}} \int_{\frac{1}{x}}^{\infty} t^n \mathrm{e}^{-t} \, \mathrm{d}t = x^n \, \mathrm{e}^{\frac{1}{x}} \, \Gamma\left(n+1, \frac{1}{x}\right),$$

and considering the relation (1.9) we obtain the following corollary.

Corollary 1.8. *With assumptions and notations of Theorem 1.5 we have*

$$\sum_{j=0}^{n} j^k\, P(n,j)\, x^j = \frac{A_k(x)}{x^k} + \frac{B_k(x)}{x^k}\, x^n\, \mathrm{e}^{\frac{1}{x}}\, \Gamma\left(n+1, \frac{1}{x}\right), \tag{1.14}$$

where $\Gamma(\alpha, z)$ *denotes the incomplete gamma function.*

2. Proofs

Proof of Theorem 1.3. Integrating by parts gives

$$\int t^r \mathrm{e}^t\, \mathrm{d}t = t^r \mathrm{e}^t - r \int t^{r-1} \mathrm{e}^t\, \mathrm{d}t.$$

Hence, for any integer $j \geqslant 1$ we obtain the recurrence

$$E_j(a) = a^j \mathrm{e}^a - j\, E_{j-1}(a).$$

Multiplying both sides by $\frac{(-1)^j}{j!}$ we rewrite this recurrence as follows

$$\frac{(-1)^j}{j!} E_j(a) - \frac{(-1)^{j-1}}{(j-1)!} E_{j-1}(a) = \frac{(-1)^j}{j!}\, a^j \mathrm{e}^a.$$

Summing over $1 \leqslant j \leqslant n$ implies

$$\frac{(-1)^n}{n!} E_n(a) - E_0(a) = \sum_{j=1}^{n} \frac{(-1)^j}{j!}\, a^j \mathrm{e}^a.$$

Since $E_0(a) = \mathrm{e}^a$, we obtain

$$\frac{(-1)^n}{n!} E_n(a) = \sum_{j=0}^{n} \frac{(-1)^j}{j!}\, a^j \mathrm{e}^a.$$

Thus,

$$E_n(a) = (-1)^n\, n!\, \mathrm{e}^a \sum_{j=0}^{n} \frac{(-1)^j}{j!}\, a^j = \mathrm{e}^a\, a^n \sum_{j=0}^{n} \frac{(-1)^j}{a^j}\, P(n,j).$$

By letting $a = -\frac{1}{x}$ for $x \neq 0$ we get (1.6). Thus,

$$\sum_{j=0}^{n} P(n,j) x^j = (-1)^n\, x^n\, \mathrm{e}^{\frac{1}{x}} \int_{-\infty}^{-\frac{1}{x}} t^n \mathrm{e}^t\, \mathrm{d}t.$$

This gives (1.6), hence concluding the proof. □

Proof of Theorem 1.1. By using the relation (1.6) with $x = 1$, for each integer $n \geqslant 0$ we obtain

$$\begin{aligned} \mathrm{e}_n\, n! = \sum_{j=0}^{n} \frac{n!}{j} = \sum_{j=0}^{n} P(n,j) &= (-1)^n\, \mathrm{e}\, E_n(-1) \\ &= (-1)^n\, \mathrm{e} \int_{-\infty}^{-1} t^n \mathrm{e}^t\, \mathrm{d}t = \mathrm{e} \int_{1}^{\infty} t^n \mathrm{e}^{-t}\, \mathrm{d}t. \end{aligned}$$

Note that $n! = \int_0^\infty t^n \mathrm{e}^{-t}\, \mathrm{d}t$. Thus,

$$\delta_n = \mathrm{e} \int_0^1 t^n \mathrm{e}^{-t}\, \mathrm{d}t \qquad (n \geqslant 0). \tag{2.1}$$

We have

$$\int_0^1 t^n \mathrm{e}^{-t}\,\mathrm{d}t = \int_0^1 t^n \sum_{j=0}^{\infty} \frac{(-t)^j}{j!}\,\mathrm{d}t = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_0^1 t^{n+j}\,\mathrm{d}t = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j+1)}.$$

For $n + c \neq 0$ we have

$$\frac{1}{n+c} = \sum_{k=1}^{r} (-1)^{k-1}\,\frac{c^{k-1}}{n^k} + \frac{(-1)^r}{n+c}\left(\frac{c}{n}\right)^r.$$

Letting $c = j + 1$ implies

$$\begin{aligned}\frac{1}{\mathrm{e}}\,\delta_n &= \sum_{j=0}^{\infty}\sum_{k=1}^{r} \frac{(-1)^{k-1}}{n^k}\,\frac{(-1)^j\,(j+1)^{k-1}}{j!} + (-1)^r \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j+1)}\left(\frac{j+1}{n}\right)^r \\ &= \sum_{k=1}^{r} \frac{(-1)^{k-1}}{n^k} \sum_{j=0}^{\infty} \frac{(-1)^j\,(j+1)^{k-1}}{j!} + \frac{(-1)^r}{n^r} \sum_{j=0}^{\infty} \frac{(-1)^j\,(j+1)^r}{j!(n+j+1)}.\end{aligned}$$

We have

$$\sum_{j=0}^{\infty} \frac{(-1)^j\,(j+1)^{k-1}}{j!} = \sum_{j=0}^{\infty} (-1)^j\,\frac{(j+1)^k}{(j+1)!} = \sum_{j=1}^{\infty} (-1)^{j-1}\,\frac{j^k}{j!} = \frac{(-1)^{k-1}}{\mathrm{e}}\,c_k,$$

with the computable constants c_k described in Remark 1.1. Also, by using Dobinski's formula (1.5) we obtain

$$\begin{aligned}\left|\sum_{j=0}^{\infty} \frac{(-1)^j\,(j+1)^r}{j!(n+j+1)}\right| &\leqslant \sum_{j=0}^{\infty} \frac{(j+1)^r}{j!(n+j+1)} \\ &< \frac{1}{n}\sum_{j=0}^{\infty} \frac{(j+1)^r}{j!} = \frac{1}{n}\sum_{j=0}^{\infty} \frac{(j+1)^{r+1}}{(j+1)!} = \frac{\mathrm{e}\,B_{r+1}}{n}.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.4. The integral representation (2.1) implies

$$\begin{aligned}\sum_{n=0}^{\infty} \delta_n^k &= \sum_{n=0}^{\infty} \left(\mathrm{e}\int_0^1 t^n \mathrm{e}^{-t}\,\mathrm{d}t\right)^k = \mathrm{e}^k \sum_{n=0}^{\infty} \prod_{j=1}^{k} \int_0^1 x_j^n \mathrm{e}^{-x_j}\,\mathrm{d}x_j \\ &= \mathrm{e}^k \sum_{n=0}^{\infty} \int_0^1 \cdots \int_0^1 \prod_{j=1}^{k} \left(x_j^n \mathrm{e}^{-x_j}\right)\mathrm{d}\mathbf{X} = \mathrm{e}^k \int_0^1 \cdots \int_0^1 \sum_{n=0}^{\infty}\prod_{j=1}^{k} \left(x_j^n \mathrm{e}^{-x_j}\right)\mathrm{d}\mathbf{X}.\end{aligned}$$

Since the function $\mathrm{e}^{-(x_1+\cdots+x_k)}$ is bounded on the hypercube region $[0,1]^k$, uniform convergence of the geometric series justifies interchange of sum and integrals. Hence, we obtain (1.7). More precisely, for $k = 2$ we have

$$\frac{1}{\mathrm{e}^2}\sum_{n=0}^{\infty} \delta_n^2 = \int_0^1 \int_0^1 \frac{\mathrm{e}^{-(x+y)}}{1-xy}\,\mathrm{d}A_{x,y} =: I,$$

say. To compute I we follow an argument due to W.J. LeVeque [6], described in Chapter 9 of [1]. We apply the change of coordinates by letting $u = \frac{y+x}{2}$ and $v = \frac{y-x}{2}$. We get the new domain of integration from old domain by first rotating it by -45° and then shrinking it by a factor of $\sqrt{2}$. This new domain of integration and the function to be integrated are symmetric with respect to the u-axis. Also, $\mathrm{d}A_{x,y} = 2\mathrm{d}A_{u,v}$. Therefore,

$$\begin{aligned} I &= 4\int_0^{\frac{1}{2}}\int_0^u \frac{\mathrm{e}^{-2u}}{1-u^2+v^2}\,\mathrm{d}v\,\mathrm{d}u + 4\int_{\frac{1}{2}}^1\int_0^{1-u} \frac{\mathrm{e}^{-2u}}{1-u^2+v^2}\,\mathrm{d}v\,\mathrm{d}u \\ &= 4\int_0^{\frac{1}{2}} \frac{\mathrm{e}^{-2u}}{\sqrt{1-u^2}}\arctan\frac{u}{\sqrt{1-u^2}}\,\mathrm{d}u + 4\int_{\frac{1}{2}}^1 \frac{\mathrm{e}^{-2u}}{\sqrt{1-u^2}}\arctan\frac{1-u}{\sqrt{1-u^2}}\,\mathrm{d}u. \end{aligned}$$

By letting $u = 1-z$ in the last integral and simplifying we obtain (1.8), hence concluding the proof. □

Proof of Theorem 1.5. We have $\frac{\mathrm{d}}{\mathrm{d}a}E_n(a) = a^n\,\mathrm{e}^a$ and $\frac{\mathrm{d}}{\mathrm{d}x}E_n(-\frac{1}{x}) = \frac{(-1)^n}{x^{n+2}}\,\mathrm{e}^{-\frac{1}{x}}$. Hence, the relation (1.6) implies

$$\frac{\mathrm{d}}{\mathrm{d}x}S_n(x) = \frac{1}{x^2} + \left(\frac{n}{x} - \frac{1}{x^2}\right)S_n(x). \tag{2.2}$$

Note that

$$\sum_{j=0}^n j\,P(n,j)\,x^j = x\,\frac{\mathrm{d}}{\mathrm{d}x}S_n(x) = \frac{1}{x} + \frac{nx-1}{x}\,S_n(x).$$

So, we obtain (1.9) for $k=1$ with $A_1(x) = 1$ and $B_1(x) = nx-1$. By induction on k, we have

$$\begin{aligned} \sum_{j=0}^n j^{k+1}\,P(n,j)\,x^j &= x\,\frac{\mathrm{d}}{\mathrm{d}x}\sum_{j=0}^n j^k\,P(n,j)\,x^j \\ &= x\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{A_k(x)}{x^k} + \frac{B_k(x)}{x^k}\,S_n(x)\right) \\ &= \frac{1}{x^{k+1}}\left(-kxA_k(x) + x^2\,\frac{\mathrm{d}}{\mathrm{d}x}A_k(x)\right) \\ &+ \frac{S_n(x)}{x^{k+1}}\left(-kxB_k(x) + x^2\,\frac{\mathrm{d}}{\mathrm{d}x}B_k(x)\right) \\ &+ \frac{1}{x^{k+1}}\left(x^2B_k(x)\,\frac{\mathrm{d}}{\mathrm{d}x}S_n(x)\right). \end{aligned}$$

Thus, by using (2.2) we obtain

$$\sum_{j=0}^n j^{k+1}\,P(n,j)\,x^j = \frac{A_{k+1}(x)}{x^k} + \frac{B_{k+1}(x)}{x^k}\,S_n(x),$$

with $A_{k+1}(x)$ and $B_{k+1}(x)$ given in (1.10). We note that $\deg A_{k+1}(x) = k$ and $\deg B_{k+1}(x) = k+1$, hence concluding the proof. □

References

[1] M. Aigner, G.M. Ziegler, *Proofs from The Book* (Sixth Edition), Springer, 2018.
[2] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products* (Seventh Edition), Academic Press, 2007.
[3] M. Hassani, *Cycles in graphs and derangements*, Math. Gaz. **88** (2004), no. 511, pp. 123–126.
[4] M. Hassani, *Derangements and applications*, J. Integer Seq. **6** (2003), no. 1, Article 03.1.2, 8 pp.
[5] M. Hassani, A. Sofo, *Sharp bounds for the constant* e, Vietnam J. Math. **43** (2015), pp. 629–633.

[6] W.J. LeVeque, *Topics in number theory (Vol. I)*, Addison-Wesley, 1956.
[7] C. Mortici, *On the sequence related to the irrationality of* e, J. Ineq. Spec. Funct. **3** (2012), no. 2, pp. 22–25.
[8] K.H. Rosen, J.G. Michaels, J.L. Gross, J.W. Grossman, D.R. Shier, *Handbook of discrete and combinatorial mathematics*, CRC Press, 2000.
[9] W. Rudin, *Principles of mathematical analysis* (Third Edition), McGraw-Hill, 1976.
[10] M.Z. Spivey, *The art of proving binomial identities*, CRC Press, 2019.
[11] The on-line encyclopaedia of integer sequences, `https://oeis.org`
[12] E.W. Weisstein, *CRC Concise Encyclopedia of Mathematics* (Second Edition), CRC Press, 2003.

Mehdi Hassani
Department of Mathematics, University of Zanjan, University Blvd., 45371-38791, Zanjan, Iran
E-mail address: `mehdi.hassani@znu.ac.ir`