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GENERALIZED FRACTIONAL OSTROWSKI TYPE INEQUALITY

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Abstract. We use Riemann-Liouville fractional integral to provide generalization of Ostrowski type inequality with bounded derivatives. Our results improved the inequalities of [19].

1. Introduction

Inequalities are one of the most powerful tools in the development of mathematics. Since the last two decades, researchers worked on fractional calculus due to its importance in inequalities. The classical form of fractional calculus is given by Bernhard Riemann and Joseph Liouville. Fractional integrals play increasingly important role on some mathematical inequalities such as Ostrowski, Grüss, Ostrowski-Grüss and Hadamard inequality etc.

We quote from [1],

"The subject of fractional calculus (that is, calculus of integrals and derivatives of an arbitrary real or complex order) was planted over 300 years ago. Since that time the fractional calculus has drawn the attention of many researchers in. In recent years, the fractional calculus has played a significant role in many areas of science and engineering."

Due to the importance of fractional integral inequalities, many authors have discussed certain generalizations of fractional integral inequalities. See the articles [21, 2, 19, 18] for recent generalizations.

In 1938, A. M. Ostrowski exhibited an inequality in his paper [16]. It is a known fact that this type of inequality could be used to estimate the deviation of functional value from its integral mean. This result nowadays known as Ostrowski inequality that can be obtained using the Montgomery identity and Lagrange's mean value theorem. In following proposition, we give this inequality from [3].

Proposition 1.1. *Let $f : I \to \mathbb{R}$ be a differentiable mappings on I^o such that $f \in L[\alpha_a, \alpha_b]$, where $\alpha_a, \alpha_b \in I$ and $\alpha_a < \alpha_b$. If $|f'(x)| \le M \ \forall \ x \in (\alpha_a, \alpha_b)$ where*

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M is positive real constant, then

$$\left|f(x)-\frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b}f(t)dt\right|\leq M(\alpha_b-\alpha_a)\left[\frac{1}{4}+\frac{\left(x-\frac{\alpha_a+\alpha_b}{2}\right)^2}{(\alpha_b-\alpha_a)^2}\right]. \tag{1.1}$$

The constant $\frac{1}{4}$ is the best possible constant that it cannot be replaced by the smaller one.

We recall the well-known Montgomery identity from "Inequalities for Functions and their Integrals and Derivatives" by D. S. Mitrinović et al. in [14].

Proposition 1.2. *let $f:[\alpha_a,\alpha_b]\to\mathbb{R}$ be an absolutely continuous function. Then*

$$f(x)=\frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b}f(t)dt+\frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b}P(x,t)f'(t)dt, \tag{1.2}$$

where Peano kernel $P(x,t)$ is given as

$$P(x,t)=\begin{cases} t-\alpha_a, & \text{if} \quad t\in[\alpha_a,x], \\ \\ t-\alpha_b, & \text{if} \quad t\in(x,\alpha_b]. \end{cases} \tag{1.3}$$

From [4] we recall Montgomery identity with parameter.

Proposition 1.3. *If $f:[\alpha_a,\alpha_b]\to\mathbb{R}$ is differentiable on $[\alpha_a,\alpha_b]$ with f' integrable on $[\alpha_a,\alpha_b]$, where $\epsilon\in[0,1]$, then generalized integrable identity holds*

$$(1-\epsilon)f(x)+\epsilon\frac{f(\alpha_a)+f(\alpha_b)}{2}-\frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b}f(t)dt=\frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b}P(x,t)f'(t)dt, \tag{1.4}$$

$\forall\ x\in\left[\alpha_a+\epsilon\frac{\alpha_b-\alpha_a}{2},\alpha_b-\epsilon\frac{\alpha_b-\alpha_a}{2}\right]$, *where $P(x,t)$ is the Peano kernel defined by*

$$P(x,t)=\begin{cases} t-\left(\alpha_a+\epsilon\frac{\alpha_b-\alpha_a}{2}\right), & \text{if} \quad t\in[\alpha_a,x], \\ \\ t-\left(\alpha_b-\epsilon\frac{\alpha_b-\alpha_a}{2}\right), & \text{if} \quad t\in(x,\alpha_b]. \end{cases} \tag{1.5}$$

For our next result we need here definition of Riemann-Liouville fractional integral from [6].

Definition 1.4. *The Riemann-Liouville fractional integral operator of order $\gamma>0$ is defined as*

$$J_{\alpha_a}^{\gamma}f(x)=\frac{1}{\Gamma(\gamma)}\int_{\alpha_a}^{x}(x-t)^{\gamma-1}f(t)dt$$
$$J_{\alpha_a}^{0}f(x)=f(x),$$

where gamma function $\Gamma(\gamma)$ is defined as

$$\Gamma(\gamma)=\int_0^\infty x^{\gamma-1}e^{-x}dx.$$

In [20] by using Riemann-Liouville fractional integrals, the authors obtained inequalities for differentiable functions that are linked with Ostrowski type inequality and discussed the following proposition to prove their results.

Proposition 1.5. *Let $f : I \to \mathbb{R}$ be differentiable mapping on I^0 with $\alpha_a, \alpha_b \in I$ $\alpha_a < \alpha_b$, $f' \in L[\alpha_a, \alpha_b]$ and for $\gamma > 1$, then Montgomery fractional identity holds*

$$f(x) = \frac{\Gamma(\gamma)}{\alpha_b - \alpha_a}(\alpha_b - x)^{1-\gamma} J^{\gamma}_{\alpha_a} f(\alpha_b) - J^{\gamma-1}_{\alpha_a}(P_1(x, \alpha_b) f(\alpha_b)) + J^{\gamma}_{\alpha_a}(P_1(x, \alpha_b) f'(\alpha_b)), \tag{1.6}$$

where $P_1(x, t)$ is the fractional Peano kernel defined by

$$P_1(x, t) = \begin{cases} \dfrac{t - \alpha_a}{\alpha_b - \alpha_a}(\alpha_b - x)^{1-\gamma}\Gamma(\gamma), & \text{if} \quad t \in [\alpha_a, x), \\ \\ \dfrac{t - \alpha_b}{\alpha_b - \alpha_a}(\alpha_b - x)^{1-\gamma}\Gamma(\gamma), & \text{if} \quad t \in [x, \alpha_b]. \end{cases} \tag{1.7}$$

Remark. *Here it is worth mentioning that Proposition* 1.3 *and Proposition* 1.5 *can be improved by using absolutely continuous functions in place of differentiable functions.*

In the next section, we propose some new results of fractional integral inequalities of Ostrowski type. First of all we establish Montgomery identity with parameters via Riemann-Lioville fractional integrals. Afterwards apply this identity to generate a lemma, which is required in our main theorem.

2. Generalized Fractional Ostrowski Type Inequalities

We need to proof the following lemmas for our main result.

Lemma 2.1. *Let $f : I \to \mathbb{R}$ be an absolutely continuous function on I^0 with $\alpha_a, \alpha_b \in I$ $\alpha_a < \alpha_b$. Then the following identity holds*

$$\begin{aligned} (1 - \epsilon) f(x) &= \frac{\Gamma(\gamma)}{\alpha_b - \alpha_a}(\alpha_b - x)^{1-\gamma} J^{\gamma}_{\alpha_a} f(\alpha_b) - J^{\gamma-1}_{\alpha_a}\left(P_2(x, \alpha_b) f(\alpha_b)\right) \\ &\quad - \frac{\epsilon(\alpha_b - x)^{1-\gamma}}{2(\alpha_b - \alpha_a)^{1-\gamma}} J^{0}_{\alpha_a} f(\alpha_a) + J^{\gamma}_{\alpha_a}\left(P_2(x, \alpha_b) f'(\alpha_b)\right), \end{aligned} \tag{2.1}$$

$\forall$ $x \in [\alpha_a, \alpha_b]$, where $\epsilon \in [0, 1]$, $\gamma > 1$ and $P_2(x, t)$ is the fractional Peano kernel defined by

$$P_2(x, t) = \begin{cases} \left[t - \left(\alpha_a + \epsilon\dfrac{\alpha_b - \alpha_a}{2}\right)\right]\dfrac{(\alpha_b - x)^{1-\gamma}}{\alpha_b - \alpha_a}\Gamma(\gamma), & \text{if} \quad t \in [\alpha_a, x), \\ \\ \left[t - \left(\alpha_b - \epsilon\dfrac{\alpha_b - \alpha_a}{2}\right)\right]\dfrac{(\alpha_b - x)^{1-\gamma}}{\alpha_b - \alpha_a}\Gamma(\gamma), & \text{if} \quad t \in [x, \alpha_b]. \end{cases} \tag{2.2}$$

Proof. Using Riemann-Liouville fractional integral operator, we get

$$\begin{aligned} & J^{\gamma}_{\alpha_a}(P_2(x, \alpha_b) f'(\alpha_b)) \\ = & \frac{1}{\Gamma(\gamma)}\int_{\alpha_a}^{\alpha_b}(\alpha_b - t)^{\gamma-1} P_2(x, t) f'(t)dt \\ = & \frac{(\alpha_b - x)^{1-\gamma}}{\alpha_b - \alpha_a}\left[\int_{\alpha_a}^{x}(\alpha_b - t)^{\gamma-1}\left(t - \left(\alpha_a + \epsilon\frac{\alpha_b - \alpha_a}{2}\right)\right) f'(t)dt \right. \\ & \left. + \int_{x}^{\alpha_b}(\alpha_b - t)^{\gamma-1}\left(t - \left(\alpha_b - \epsilon\frac{\alpha_b - \alpha_a}{2}\right)\right) f'(t)dt\right] \end{aligned}$$

$$\begin{aligned}
&= (1-\epsilon)f(x) - \frac{(\alpha_b - x)^{1-\gamma}}{(\alpha_b-\alpha_a)}\int_{\alpha_a}^{\alpha_b}(\alpha_b-t)^{\gamma-1}f(t)dt \\
&\quad + \frac{\epsilon(\alpha_b-x)^{1-\gamma}}{2(\alpha_b-\alpha_a)^{1-\gamma}}f(\alpha_a) + \frac{\gamma-1}{\Gamma(\gamma)}\int_{\alpha_a}^{\alpha_b}(\alpha_b-t)^{\gamma-2}P_2(x,t)f(t)dt \\
&= (1-\epsilon)f(x) - \frac{(\alpha_b - x)^{1-\gamma}}{(\alpha_b-\alpha_a)}\Gamma(\gamma)J_{\alpha_a}^{\gamma}f(\alpha_b) \\
&\quad + \frac{\epsilon(\alpha_b-x)^{1-\gamma}}{2(\alpha_b-\alpha_a)^{1-\gamma}}J_{\alpha_a}^{0}f(\alpha_a) + J_{\alpha_a}^{\gamma-1}(P_2(x,\alpha_b)f(\alpha_b)).
\end{aligned}$$

After rearranging the terms, we get (2.1). □

Remark. *If we replace* $\gamma = 1$ *in* (2.1)*, then we get the identity* (1.4)*.*

Remark. *If we replace* $\epsilon = 0$ *in* (2.1)*, then we get the identity* (1.6)*.*

Lemma 2.2. *Let all suppositions of Lemma* 2.1 *be valid. Then following identity holds*

$$\begin{aligned}
&(1-\epsilon)f(x) \\
=&2\Gamma(\gamma)\frac{(\alpha_b-x)^{1-\gamma}}{\alpha_b-\alpha_a}J_{\alpha_a}^{\gamma}f(\alpha_b) - J_{\alpha_a}^{\gamma-1}\left(P_2(x,\alpha_b)f(\alpha_b)\right) \\
&-\frac{(\alpha_b-x)^{1-\gamma}}{\alpha_b-\alpha_a}\Gamma(\gamma)J_{\alpha_a}^{\gamma-1}\left((\alpha_b-x)f(\alpha_b)\right) + 2J_{\alpha_a}^{\gamma}\left(K_1(x,\alpha_b)f'(\alpha_b)\right) \\
&-\frac{(\alpha_b-x)^{1-\gamma}}{(\alpha_b-\alpha_a)^{2-\gamma}}\left(x-\alpha_a+\epsilon\frac{\alpha_b-\alpha_a}{2}\right)J_{\alpha_a}^{0}f(\alpha_a),
\end{aligned} \tag{2.3}$$

$\forall\ x \in [\alpha_a, \alpha_b]$*, where* $\epsilon \in [0,1]$*,* $\gamma > 1$ *and* $K_1(x,t)$ *is the fractional Peano kernel defined by*

$$K_1(x,t) = \begin{cases} \left[t - \left(\frac{\alpha_a + x}{2} + \epsilon\frac{\alpha_b-\alpha_a}{4}\right)\right]\frac{(\alpha_b-x)^{1-\gamma}}{\alpha_b-\alpha_a}\Gamma(\gamma), & \textit{if} \quad t\in[\alpha_a, x), \\ \left[t - \left(\frac{\alpha_b + x}{2} - \epsilon\frac{\alpha_b-\alpha_a}{4}\right)\right]\frac{(\alpha_b-x)^{1-\gamma}}{\alpha_b-\alpha_a}\Gamma(\gamma), & \textit{if} \quad t\in[x, \alpha_b], \end{cases}$$

Proof. Using Riemann-Liouville fractional integral operator on $K_1(x,t)$, we get

$$\begin{aligned}
&J_{\alpha_a}^{\gamma}(K_1(x,\alpha_b)f'(\alpha_b)) \\
=& \frac{1}{\Gamma(\gamma)}\int_{\alpha_a}^{\alpha_b}(\alpha_b-t)^{\gamma-1}K_1(x,t)f'(t)dt \\
=& \frac{(\alpha_b-x)^{1-\gamma}}{\alpha_b-\alpha_a}\left[\int_{\alpha_a}^{x}(\alpha_b-t)^{\gamma-1}\left(t-\left(\frac{\alpha_a+x}{2}+\epsilon\frac{\alpha_b-\alpha_a}{4}\right)\right)f'(t)dt\right. \\
&\left. + \int_{x}^{\alpha_b}(\alpha_b-t)^{\gamma-1}\left(t-\left(\frac{\alpha_b+x}{2}-\epsilon\frac{\alpha_b-\alpha_a}{4}\right)\right)f'(t)dt\right],
\end{aligned}$$

after some computations and using the concept of fractional calculus, we get

$$\begin{aligned}
&J_{\alpha_a}^{\gamma}(K_1(x,\alpha_b)f'(\alpha_b)) \\
=&\frac{1}{2}\left[J_{\alpha_a}^{\gamma}(P_2(x,\alpha_b)f'(\alpha_b)) + \frac{(\alpha_b-x)^{1-\gamma}}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b}(\alpha_b-t)^{\gamma-1}(t-x)f'(t)dt\right].
\end{aligned} \tag{2.4}$$

We also have

$$\int_{\alpha_a}^{\alpha_b} (\alpha_b - t)^{\gamma-1}(t-x)f'(t)dt$$
$$= (x-\alpha_a)(\alpha_b-\alpha_a)^{\gamma-1} J^0_{\alpha_a} f(\alpha_a)$$
$$+ \Gamma(\gamma) J^{\gamma-1}_{\alpha_a}\left((\alpha_b - x)f(\alpha_b)\right) - \Gamma(\gamma) J^{\gamma}_{\alpha_a} f(\alpha_b). \tag{2.5}$$

Now from Lemma 2.1, we have

$$J^{\gamma}_{\alpha_a}\left(P_2(x,\alpha_b)f'(\alpha_b)\right)$$
$$= (1-\epsilon)f(x) - \frac{\Gamma(\gamma)}{\alpha_b-\alpha_a}(\alpha_b - x)^{1-\gamma} J^{\gamma}_{\alpha_a} f(\alpha_b)$$
$$+ J^{\gamma-1}_{\alpha_a}\left(P_2(x,\alpha_b)f(\alpha_b)\right) + \frac{\epsilon(\alpha_b - x)^{1-\gamma}}{2(\alpha_b-\alpha_a)^{1-\gamma}} J^0_{\alpha_a} f(\alpha_a). \tag{2.6}$$

Using (2.5) and (2.6) in (2.4), we get the result

$$J^{\gamma}_{\alpha_a}(K_1(x,\alpha_b)f'(\alpha_b))$$
$$= \frac{1}{2}(1-\epsilon)f(x) - \Gamma(\gamma)\frac{(\alpha_b-x)^{1-\gamma}}{(\alpha_b-\alpha_a)} J^{\gamma}_{\alpha_a} f(\alpha_b)$$
$$+ \frac{1}{2} J^{\gamma-1}_{\alpha_a}(P_2(x,\alpha_b)f(\alpha_b)) + \frac{(\alpha_b - x)^{1-\gamma}}{2(\alpha_b-\alpha_a)^{2-\gamma}} J^0_{\alpha_a} f(\alpha_a)\left(x - \left(\alpha_a - \epsilon\frac{\alpha_b-\alpha_a}{2}\right)\right)$$
$$+ \frac{(\alpha_b-\alpha_a)^{1-\gamma}}{2(\alpha_b-\alpha_a)}\Gamma(\gamma) J^{\gamma-1}_{\alpha_a}\left((\alpha_b - x)f(\alpha_b)\right),$$

which yields the required result. □

Remark. *If we replace $\gamma = 1$ in (2.3), then we obtain the following corollary.*

Corollary 2.3. *Let all suppositions of Lemma 2.1 be valid. Then*

$$\frac{1}{2}(1-\epsilon)f(x)$$
$$= \frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b} f(t)dt + \frac{1}{2(\alpha_b-\alpha_a)}\left[f(\alpha_b)\left(x - \left(\alpha_b + \epsilon\frac{\alpha_b-\alpha_a}{2}\right)\right)\right]$$
$$- f(\alpha_a)\left(x - \left(\alpha_a - \epsilon\frac{\alpha_b-\alpha_a}{2}\right)\right) + \frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b} K_2(x,t)f'(t)dt, \tag{2.7}$$

$\forall\ x \in [\alpha_a, \alpha_b]$ and where $\epsilon \in [0,1]$ and

$$K_2(x,t) = \begin{cases} \left[t - \left(\dfrac{\alpha_a + x}{2} + \epsilon\dfrac{\alpha_b-\alpha_a}{4}\right)\right], & \text{if} \quad t \in [\alpha_a, x), \\[2ex] \left[t - \left(\dfrac{\alpha_b + x}{2} - \epsilon\dfrac{\alpha_b-\alpha_a}{4}\right)\right], & \text{if} \quad t \in [x, \alpha_b]. \end{cases}$$

Remark. *If we replace $\epsilon = 0$ in (2.3), then we obtain the following corollary.*

Corollary 2.4. *Let all suppositions of Lemma* 2.1 *be valid. Then*

$$\begin{aligned} f(x) &= 2\Gamma(\gamma)\frac{(\alpha_b-x)^{1-\gamma}}{\alpha_b-\alpha_a}J^{\gamma}_{\alpha_a}f(\alpha_b)-J^{\gamma-1}_{\alpha_a}\left(P_1(x,\alpha_b)f(\alpha_b)\right) \\ &-\frac{(\alpha_b-x)^{1-\gamma}}{\alpha_b-\alpha_a}\Gamma(\gamma)J^{\gamma-1}_{\alpha_a}\left((\alpha_b-x)f(\alpha_b)\right)+2J^{\gamma}_{\alpha_a}\left(K_3(x,\alpha_b)f'(\alpha_b)\right) \\ &-\frac{(\alpha_b-x)^{1-\gamma}}{(\alpha_b-\alpha_a)^{2-\gamma}}\left(x-\alpha_a\right)J^{0}_{\alpha_a}f(\alpha_a), \end{aligned} \tag{2.8}$$

$\forall\ x\in[\alpha_a,\alpha_b]$, *where* $\gamma>1$ *and*

$$K_3(x,t)=\begin{cases} \left(t-\dfrac{\alpha_a+x}{2}\right)\dfrac{(\alpha_b-x)^{1-\gamma}}{\alpha_b-\alpha_a}\Gamma(\gamma), & \text{if} \quad t\in[\alpha_a,x), \\ \left(t-\dfrac{\alpha_b+x}{2}\right)\dfrac{(\alpha_b-x)^{1-\gamma}}{\alpha_b-\alpha_a}\Gamma(\gamma), & \text{if} \quad t\in[x,\alpha_b], \end{cases}$$

and $P_1(x,t)$ *is defined in* (1.7).

Remark. *If we replace* $\gamma=1$ *in* (2.8), *then we obtain the following corollary.*

Corollary 2.5. *Let all suppositions of Lemma* 2.1 *be valid. Then*

$$\begin{aligned} \frac{1}{2}f(x) &= \frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b}f(t)dt+\frac{(x-\alpha_b)f(\alpha_b)-(x-\alpha_a)f(\alpha_a)}{2(\alpha_b-\alpha_a)} \\ &+\frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b}K_4(x,t)f'(t)dt, \end{aligned}$$

$\forall\ x\in[\alpha_a,\alpha_b]$, *where*

$$K_4(x,t)=\begin{cases} \left(t-\dfrac{\alpha_a+x}{2}\right), & \text{if} \quad t\in[\alpha_a,x), \\ \left(t-\dfrac{\alpha_b+x}{2}\right), & \text{if} \quad t\in[x,\alpha_b], \end{cases}$$

which is cited in [23] *by F. Tong and L. Guan.*

By using Lemma 2.2, we obtain generalized Ostrowski-Grüss fractional integral inequality in the next theorem.

Theorem 2.6. *Let* $f:I\to\mathbb{R}$ *be an absolutely continuous function on* I^o *such that* $\alpha_a,\alpha_b\in I$ *and* $\alpha_a<\alpha_b$. *If* $|f'(x)|\leq M$ *a.e.* $\forall\ x\in(\alpha_a,\alpha_b)$ *where* M *is positive real constant, then the following inequality holds*

$$\Bigg|\frac{1}{2}(1-\epsilon)f(x)-2\Gamma(\gamma)\frac{(\alpha_b-x)^{1-\gamma}}{2(\alpha_b-\alpha_a)}J^{\gamma}_{\alpha_a}f(\alpha_b)+\frac{(\alpha_b-x)^{1-\gamma}}{2(\alpha_b-\alpha_a)}\Gamma(\gamma)J^{\gamma-1}_{\alpha_a}\left((\alpha_b-x)f(\alpha_b)\right)$$
$$+\frac{1}{2}J^{\gamma-1}_{\alpha_a}(P_2(x,\alpha_b)f(\alpha_b))+\frac{(\alpha_b-x)^{1-\gamma}}{2(\alpha_b-\alpha_a)^{2-\gamma}}J^{0}_{\alpha_a}f(\alpha_a)\left(x-\alpha_a+\epsilon\frac{\alpha_b-\alpha_a}{2}\right)\Bigg|$$

$$
\begin{aligned}
&\le \frac{M(\alpha_b - x)^{1-\gamma}}{\alpha_b - \alpha_a} \\
&\quad \times \left[\frac{(x-\alpha_a)}{2\gamma} \left\{ (\alpha_b - \alpha_a)^{\gamma} - (\alpha_b - x)^{\gamma} + (\alpha_b - x)^{\gamma+1} + \frac{\epsilon}{2}(\alpha_b - \alpha_a)^{\gamma+1} \right\} \right. \\
&\quad + \frac{1}{\gamma(\gamma+1)} \left\{ 2\left(\alpha_b - \frac{\alpha_a + x}{2} + \epsilon \frac{\alpha_b - \alpha_a}{4} \right)^{\gamma+1} - 2(\alpha_b - x)^{\gamma+1} - (\alpha_b - \alpha_a)^{\gamma+1} \right. \\
&\quad \left. \left. + 2\left(\frac{\alpha_b - x}{2} - \epsilon \frac{\alpha_b - \alpha_a}{4} \right)^{\gamma+1} \right\} \right], \qquad (2.9)
\end{aligned}
$$

$\forall\ x \in [\alpha_a, \alpha_b]$, where $\epsilon \in [0,1]$ and $\gamma > 1$.

Proof. From Lemma 2.2, consider

$$
\begin{aligned}
& I \\
=& \left| \frac{1}{2}(1-\epsilon) f(x) - 2\Gamma(\gamma) \frac{(\alpha_b - x)^{1-\gamma}}{2(\alpha_b - \alpha_a)} J^{\gamma}_{\alpha_a} f(\alpha_b) + \frac{(\alpha_b - x)^{1-\gamma}}{2(\alpha_b - \alpha_a)} \Gamma(\gamma) J^{\gamma-1}_{\alpha_a} \left((\alpha_b - x) f(\alpha_b) \right) \right. \\
& \left. + \frac{1}{2} J^{\gamma-1}_{\alpha_a} \left(P_2(x, \alpha_b) f(\alpha_b) \right) + \frac{(\alpha_b - x)^{1-\gamma}}{2(\alpha_b - \alpha_a)^{2-\gamma}} \left(x - \alpha_a + \epsilon \frac{\alpha_b - \alpha_a}{2} \right) J^{0}_{\alpha_a} f(\alpha_a) \right| \\
=& \left| J^{\gamma}_{\alpha_a} \left(K_1(x, \alpha_b) f'(\alpha_b) \right) \right| \\
=& \left| \frac{1}{\Gamma(\gamma)} \int_{\alpha_a}^{\alpha_b} (\alpha_b - t)^{\gamma-1} K_1(x,t) f'(t) dt \right| \\
\le& \frac{1}{\Gamma(\gamma)} \int_{\alpha_a}^{\alpha_b} (\alpha_b - t)^{\gamma-1} \left| K_1(x,t) \right| \left| f'(t) \right| dt \\
\le& \frac{M}{\Gamma(\gamma)} \int_{\alpha_a}^{\alpha_b} (\alpha_b - t)^{\gamma-1} \left| K_1(x,t) \right| dt \\
=& M \frac{(\alpha_b - x)^{1-\gamma}}{\alpha_b - \alpha_a} \left[\int_{\alpha_a}^{x} (\alpha_b - t)^{\gamma-1} \left| t - \left(\frac{\alpha_a + x}{2} + \epsilon \frac{\alpha_b - \alpha_a}{4} \right) \right| dt \right. \\
& \left. + \int_{x}^{\alpha_b} (\alpha_b - t)^{\gamma-1} \left| t - \left(\frac{\alpha_b + x}{2} - \epsilon \frac{\alpha_b - \alpha_a}{4} \right) \right| dt \right] \\
=& M \frac{(\alpha_b - x)^{1-\gamma}}{\alpha_b - \alpha_a} \left[I_1 + I_2 + I_3 + I_4 \right], \qquad (2.10)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int_{\alpha_a}^{\frac{\alpha_a + x}{2} + \epsilon \frac{\alpha_b - \alpha_a}{4}} (\alpha_b - t)^{\gamma-1} \left\{ \left(\frac{\alpha_a + x}{2} + \epsilon \frac{\alpha_b - \alpha_a}{4} \right) - t \right\} dt \\
&= \left(\frac{x - \alpha_a}{2} + \epsilon \frac{\alpha_b - \alpha_a}{4} \right) \frac{(\alpha_b - \alpha_a)^{\gamma}}{\gamma} + \frac{1}{\gamma(\gamma+1)} \left(\alpha_b - \frac{\alpha_a + x}{2} - \epsilon \frac{\alpha_b - \alpha_a}{4} \right)^{\gamma+1} \\
&\quad - \frac{(\alpha_b - \alpha_a)^{\gamma+1}}{\gamma(\gamma+1)}, \qquad (2.11)
\end{aligned}
$$

$$I_2 = \int_{\frac{\alpha_a+x}{2}+\epsilon\frac{\alpha_b-\alpha_a}{4}}^{x} (\alpha_b - t)^{\gamma-1}\left\{t - \left(\frac{\alpha_a + x}{2} + \epsilon\frac{\alpha_b-\alpha_a}{4}\right)\right\}dt$$
$$= \left(\frac{\alpha_a - x}{2} + \epsilon\frac{\alpha_b-\alpha_a}{4}\right)\frac{(\alpha_b - x)^\gamma}{\gamma} + \frac{1}{\gamma(\gamma+1)}\left(\alpha_b - \frac{\alpha_a+x}{2} - \epsilon\frac{\alpha_b-\alpha_a}{4}\right)^{\gamma+1}$$
$$- \frac{(\alpha_b - x)^{\gamma+1}}{\gamma(\gamma+1)}, \tag{2.12}$$

$$I_3 = \int_{x}^{\frac{\alpha_b+x}{2}-\epsilon\frac{\alpha_b-\alpha_a}{4}} (\alpha_b - t)^{\gamma-1}\left\{\left(\frac{\alpha_b + x}{2} - \epsilon\frac{\alpha_b-\alpha_a}{4}\right)\right\}dt$$
$$= \left(\frac{\alpha_b - x}{2} - \epsilon\frac{\alpha_b-\alpha_a}{4}\right)\frac{(\alpha_b - x)^\gamma}{\gamma} + \frac{1}{\gamma(\gamma+1)}\left(\frac{\alpha_b - x}{2} + \epsilon\frac{\alpha_b-\alpha_a}{4}\right)^{\gamma+1}$$
$$- \frac{(\alpha_b - x)^{\gamma+1}}{\gamma(\gamma+1)} \tag{2.13}$$

and

$$I_4 = \int_{\frac{\alpha_b+x}{2}-\epsilon\frac{\alpha_b-\alpha_a}{4}}^{\alpha_b} (\alpha_b - t)^{\gamma-1}\left\{t - \left(\frac{\alpha_b + x}{2} - \epsilon\frac{\alpha_b-\alpha_a}{4}\right)\right\}dt$$
$$= \frac{1}{\gamma(\gamma+1)}\left(\frac{\alpha_b - x}{2} - \epsilon\frac{\alpha_b-\alpha_a}{4}\right)^{\gamma+1}. \tag{2.14}$$

Using (2.11) – (2.14) in (2.10), we get the bound

$$I \leq \frac{M(\alpha_b - x)^{1-\gamma}}{\alpha_b - \alpha_a}$$
$$\times\left[\frac{(x-\alpha_a)}{2\gamma}\left\{(\alpha_b-\alpha_a)^\gamma - (\alpha_b - x)^\gamma + (\alpha_b - x)^{\gamma+1} + \frac{\epsilon}{2}(\alpha_b-\alpha_a)^{\gamma+1}\right\}\right.$$
$$+\frac{1}{\gamma(\gamma+1)}\left\{2\left(\alpha_b - \frac{\alpha_a+x}{2} + \epsilon\frac{\alpha_b-\alpha_a}{4}\right)^{\gamma+1} - 2(\alpha_b - x)^{\gamma+1} - (\alpha_b-\alpha_a)^{\gamma+1}\right.$$
$$\left.\left.+2\left(\frac{\alpha_b - x}{2} - \epsilon\frac{\alpha_b-\alpha_a}{4}\right)^{\gamma+1}\right\}\right].$$

□

Remark. *If we replace $\gamma = 1$ in (2.9), then we obtain the following corollary.*

Corollary 2.7. *Let all suppositions of Theorem 2.6 be valid. Then*

$$\left|\frac{1}{2}(1-\epsilon)f(x) - \frac{1}{2(\alpha_b-\alpha_a)}\left\{\left(x - \left(\alpha_b + \epsilon\frac{\alpha_b-\alpha_a}{2}\right)\right)f(\alpha_b)\right.\right.$$
$$\left.\left.-\left(x - \left(\alpha_a - \epsilon\frac{\alpha_b-\alpha_a}{2}\right)\right)f(\alpha_a)\right\} - \frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b} f(t)dt\right|$$
$$\leq \frac{M}{4(\alpha_b-\alpha_a)}\left[(x-\alpha_a)^2 + (\alpha_b - x)^2 + \frac{\epsilon(\alpha_b-\alpha_a)^2}{4}\right], \tag{2.15}$$

$\forall\ x \in [\alpha_a, \alpha_b]$, *where* $\epsilon \in [0, 1]$.

Remark. *If we replace $\epsilon = 0$ in (2.9), then we obtain the following corollary.*

Corollary 2.8. *Let all suppositions of Theorem* 2.6 *be valid. Then*

$$\begin{aligned}
&\left|\frac{1}{2}f(x) - 2\Gamma(\gamma)\frac{(\alpha_b - x)^{1-\gamma}}{2(\alpha_b-\alpha_a)}J^{\gamma}_{\alpha_a}f(\alpha_b) + \frac{1}{2}J^{\gamma-1}_{\alpha_a}(P_1(x,\alpha_b)f(\alpha_b))\right.\\
&\left.+\frac{(\alpha_b - x)^{1-\gamma}}{2(\alpha_b-\alpha_a)}\Gamma(\gamma)J^{\gamma-1}_{\alpha_a}\left((\alpha_b-x)f(\alpha_b)\right) + \frac{(\alpha_b - x)^{1-\gamma}}{2(\alpha_b-\alpha_a)^{2-\gamma}}J^{0}_{\alpha_a}f(\alpha_a)\,(x-\alpha_a)\right|\\
\leq&\frac{M(\alpha_b - x)^{1-\gamma}}{\alpha_b-\alpha_a}\left[\frac{(x-\alpha_a)}{2\gamma}\left\{(\alpha_b-\alpha_a)^{\gamma} - (\alpha_b-x)^{\gamma} + (\alpha_b-x)^{\gamma+1}\right\} + \frac{1}{\gamma(\gamma+1)}\right.\\
&\left.\times\left\{2\left(\alpha_b - \frac{\alpha_a+x}{2}\right)^{\gamma+1} - 2(\alpha_b-x)^{\gamma+1} - (\alpha_b-\alpha_a)^{\gamma+1} \;\; +2\left(\frac{\alpha_b-x}{2}\right)^{\gamma+1}\right\}\right],
\end{aligned} \tag{2.16}$$

$\forall\ x \in [\alpha_a, \alpha_b]$, *where* $\gamma > 1$.

Remark. *Let either* $\epsilon = 0$ *in* (2.15) *or* $\gamma = 1$ *in* (2.16). *Then we obtain the following corollary.*

Corollary 2.9. *Let all suppositions of Theorem* 2.6 *be valid. Then*

$$\begin{aligned}
&\left|\frac{1}{2}f(x) - \frac{1}{2(\alpha_b-\alpha_a)}\left\{(x-\alpha_b)\,f(\alpha_b) - (x-\alpha_a)f(\alpha_a)\right\} - \frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b} f(t)dt\right|\\
\leq&\frac{M}{4(\alpha_b-\alpha_a)}\left[(x-\alpha_a)^2 + (\alpha_b-x)^2\right],
\end{aligned} \tag{2.17}$$

$\forall\ x \in [\alpha_a, \alpha_b]$.

Remark. *Let* $x = \dfrac{\alpha_a+\alpha_b}{2}$ *in* (2.15). *Then we obtain the following corollary*

Corollary 2.10. *Let all suppositions of Theorem* 2.6 *be valid. Then*

$$\begin{aligned}
&\left|\frac{1}{2}(1-\epsilon)f\left(\frac{\alpha_a+\alpha_b}{2}\right) + (1+\epsilon)\frac{f(\alpha_a)+f(\alpha_b)}{4} - \frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b} f(t)dt\right|\\
\leq\;& \frac{M}{16}(\alpha_b-\alpha_a)(2+\epsilon),
\end{aligned} \tag{2.18}$$

where $\epsilon \in [0, 1]$.

Remark. *Let* $\epsilon = 0$ *in* (2.18). *Then we get the bound for average midpoint and trapezoidal inequality in the following corollary.*

Corollary 2.11. *Let all suppositions of Theorem* 2.6 *be valid. Then*

$$\begin{aligned}
&\left|\frac{1}{2}f\left(\frac{\alpha_a+\alpha_b}{2}\right) + \frac{f(\alpha_a)+f(\alpha_b)}{4} - \frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b} f(t)dt\right|\\
\leq\;& \frac{M}{8}(\alpha_b-\alpha_a).
\end{aligned}$$

Remark. *Let* $\epsilon = \dfrac{1}{2}$ *in* (2.18). *Then we get the bound for perturbed trapezoidal inequality in the following corollary.*

Corollary 2.12. *Let all suppositions of Theorem* 2.6 *be valid. Then*

$$\begin{aligned}
&\left|\frac{1}{4}f\left(\frac{\alpha_a+\alpha_b}{2}\right) + \frac{3}{8}\Big(f(\alpha_a)+f(\alpha_b)\Big) - \frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b} f(t)dt\right|\\
\leq\;& \frac{5}{32}(\alpha_b-\alpha_a)M.
\end{aligned}$$

Remark. *Let* $\epsilon = \frac{1}{3}$ *in* (2.18). *Then we get the bound for perturbed trapezoidal inequality in the following corollary.*

Corollary 2.13. *Let all suppositions of Theorem* 2.6 *be valid. Then*

$$\left|\frac{1}{3}f\left(\frac{\alpha_a+\alpha_b}{2}\right)+\frac{1}{3}\Big(f(\alpha_a)+f(\alpha_b)\Big)-\frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b} f(t)dt\right|$$
$$\leq \frac{7}{48}(\alpha_b-\alpha_a)M.$$

Remark. *Let* $\epsilon = \frac{1}{4}$ *in* (2.18). *Then we get the bound for perturbed trapezoidal inequality in the following corollary.*

Corollary 2.14. *Let all suppositions of Theorem* 2.6 *be valid. Then*

$$\left|\frac{3}{8}f\left(\frac{\alpha_a+\alpha_b}{2}\right)+\frac{5}{16}\Big(f(\alpha_a)+f(\alpha_b)\Big)-\frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b} f(t)dt\right|$$
$$\leq \frac{9}{64}(\alpha_b-\alpha_a)M.$$

Remark. *Let* $x = \alpha_a$ *or* $x = \alpha_b$ *in* (2.17). *Then we get the bound for trapezoidal inequality (also Hermite-hadamard right bound) in the following corollary.*

Corollary 2.15. *Let all suppositions of Theorem* 2.6 *be valid. Then*

$$\left|\frac{f(\alpha_a)+f(\alpha_b)}{2}-\frac{1}{\alpha_b-\alpha_a}\int_{\alpha_a}^{\alpha_b} f(t)dt\right| \leq \frac{M}{4}(\alpha_b-\alpha_a).$$

3. Conclusion

In this article our target was to generalize the results of [19]. We have obtained fractional Ostrowski inequality with bounded derivatives by using the Riemann-Liouville integral. By using appropriate substitution we got previous results as well as some better bounds stated in [19].

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