q-LAPLACE TRANSFORM FOR PRODUCT OF GENERAL CLASS OF q-POLYNOMIALS AND q-ANALOGUE OF I-FUNCTION

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Abstract. In this paper, the basic analogue of the Laplace transforms involving the product of a general class of q-polynomials along with q-analogue of Fox’s H-function and q-analogue of I-function is evaluated. Limiting cases of the main outcomes are also evaluated. The paper shows a large variety of outcomes that can be achieved.

1. Introduction

In the concept of q-calculus, various notable integral transforms have the equivalent q-analogues, which includes the basic analogue of Laplace transforms [12], [1], the basic analogue of Sumudu transforms [2]-[4], the q-Wavelet transform [8], the q-Mellin transform [9], q-Mangontarum transforms [5], q-natural transforms [6], and many more, see also [7]. These q-integral transforms plays an important role for solving the q-fractional differential and integral equations; particularly see [10] and [16].

Investigation of integral transforms (including, q-transforms) image formulas with different special functions find important significance and applications in specific sub-fields of applied mathematical research. Therefore, a number of workers including, Albayrak et al. [3], Al-Omari [5], [6], Purohit and Ucar [16], Yadav and Purohit [24]-[26] etc., have studied the property, applications and evaluated a range of image formulas comprising q-special functions. Also, the degenerate Laplace transforms with the degenerate gamma function found in the papers [13, 14]. Recently, Vyas et al. [23] have found the q-Sumudu image formulas for a product of q-polynomials along with basic (or q-)analogue of Fox’s H-function and I-functions. Motivated by the fascinating result, we further investigate the possibility of evaluating the q-Laplace transforms and generalized q-special functions for the family product of q-polynomials.

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We need some preliminaries for our investigation related to the $q$-calculus, which we add in this section.

Hahn [12] characterized the basic analogues of the Laplace transform

$$\varphi(p) = \int_0^\infty e^{-pt} f(p) \, d_q t \quad (p > 0),$$

(1.1)

by the subsequent $q$-integrals

$$qL_p \{ f(t) \} = \frac{1}{1-q} \int_0^{p^{-1}} E_q(qpt) f(t) \, d_q t$$

(1.2)

and

$$qL_p \{ f(t) \} = \frac{1}{1-q} \int_0^\infty e_{q}(-pt) f(t) \, d_q t,$$

(1.3)

where $\Re(p) > 0$ and the $q$-exponential series are as follow

$$e_q(t) = \sum_{m=0}^\infty \frac{t^m}{(q; q)_m},$$

(1.4)

and

$$E_q(t) = \sum_{m=0}^\infty \frac{(-1)^m q^{m(m-1)/2} t^m}{(q; q)_m}.$$  

(1.5)

Following Gasper and Rahman [11], the $q$-integration is given by

$$\int_0^{y} f(z) \, d(z; q) = y (1-q) \sum_{k=0}^{\infty} q^k f(xq^k).$$

(1.6)

By the use of result (1.6), operator (1.3) can be articulated as

$$\varphi(p) \equiv qL_p \{ f(t) \} = \frac{(q; q)_\infty}{p} \sum_{j=0}^{\infty} \frac{q^j f(p^{-1} q^j)}{(q; q)_j}.$$  

(1.7)

The statement set out in (1.3) and (1.7) shall be characteristically addressed by

$$f(z) \supset_q \varphi(p),$$

wherein the function $f(z)$ is known as the main function, while the function $\varphi(p)$ is representing the $q$-Laplace image for the main function $f(z)$.

If $\beta$ is real or else complex, and $|q| < 1$, the $q$-analogues of shifted factorial, binomial expansion and gamma function are characterized by

$$(\beta; q)_n = \begin{cases} 1, & \text{if } n = 0 \\ (1-\beta) (1-\beta q) \cdots (1-\beta q^{n-1}), & \text{if } n \in \mathbb{N}, \end{cases}$$

(1.8)

and

$$(x-y)_\nu = x^\nu \prod_{n=0}^{\infty} \left[ \frac{1 - (y/x) q^n}{1 - (y/x) q^{\nu+n}} \right],$$

(1.9)

and

$$\Gamma_q(\beta) = \frac{(q; q)_\infty (1-q)^{1-\beta}}{(q^\beta; q)_\infty} = \frac{(1-q)^{\beta-1}}{(1-q)^{\beta-1}} = \frac{(q; q)^{\beta-1}}{(1-q)^{\beta-1}},$$

(1.10)

where $\beta \neq 0, -1, -2, \ldots$. 

Presently, we lead by thinking back on an arrangement of $q$-polynomials $f_{n,N}(x; q)$ in expressions of a bounded complex sequence $\{S_{j,q}\}_{n=0}^{\infty}$, given as (cf. Srivastava and Agarwal [21])

$$f_{n,N}(x, q) = \sum_{j=0}^{[n/N]} \left[ \frac{n}{Nj} \right] S_{j,q} x^j \quad (n = 0, 1, 2...)$$  \hspace{1cm} (1.11)

where $N$ to be any positive integer.

Saxena and Kumar [17] derived a basic (or $q$-) analogue for the $I$-function in form of the Mellin-Barnes kind $q$-contour integral as:

$$I_{m_1,n_1}^{m_1,n_1} \left[ x ; q \right] \left( a_j, \alpha_j \right)_1, n, \left( a_{ji}, \alpha_{ji} \right)_{n+1, A_1}, \left( b_j, \beta_j \right)_1, m, \left( b_{ji}, \beta_{ji} \right)_{m+1, B_1} = \frac{1}{2\pi \omega} \int_{C} \Phi(s) x^s d_q s,$$  \hspace{1cm} (1.12)

where

$$\Phi(s) = \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j}) \prod_{j=1}^{n_1} G(q^{1-a_j+\alpha_j})}{\sum_{i=1}^{r} \left\{ \prod_{j=m_1+1}^{B_1} G(q^{1-b_j-\beta_j}) \prod_{j=n_1+1}^{A_1} G(q^{s_j-\alpha_j}) \right\} G(q^1) \sin \pi s},$$

and

$$G(q^\delta) = \left\{ \prod_{n=0}^{\infty} (1 - q^{\delta+n}) \right\}^{-1} = \frac{1}{(q^\delta; q)_\infty},$$

provided $0 \leq m_1 \leq B_1; 0 \leq n_1 \leq A_1$; $i = 1, 2, \cdots, r$; and $r$ is finite; $\omega = \sqrt{-1}$.

Moreover $a_j, b_j, a_{ji}, b_{ji}$ are complex and $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ be real and positive numbers.

The curve of integration $C$ run from $-i\infty$ to $+i\infty$ within such a way that every poles of $G(q^{b_j-\beta_j})$; $1 \leq j \leq m_1$, are to its right side, and those of $G(q^{1-a_j+\alpha_j})$, $1 \leq j \leq n_1$, are to its left and at least some $\varepsilon > 0$ distance away from the contour $C$. The $q$-integral converge for $Re[s \log(x) - \log \sin \pi s] < 0$, if huge value of $|s|$ on the contour, to facilitate if $|\arg x| < \pi$. It can be observed that other suitably indented contours parallel to the imaginary axis will replace the contour of integration $C$.

It is remarkable to view that for $A_1 = A; B_1 = B; r = 1$, equation (1.12) yields $q$-extension for Fox’s $H$-function given by Saxena et al. [20],

$$H_{m_1,n_1}^{m_1,n_1} \left[ x ; q \right] \left( a, \alpha \right)_1 \left( b, \beta \right) = \frac{1}{2\pi \omega} \int_{C} \phi(s) x^s d_q s,$$  \hspace{1cm} (1.13)

where

$$\phi(s) = \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j}) \prod_{j=1}^{n_1} G(q^{1-a_j+\alpha_j})}{\prod_{j=m_1+1}^{B} G(q^{1-b_j-\beta_j}) \prod_{j=n_1+1}^{A} G(q^{a_j-\alpha_j})G(q^1) \sin \pi s}.$$

If we put $\alpha_i = \beta_i = 1, \forall i$ and $j$ in the equation (1.13), thereupon it reduce to a $q$-analogue for Meijer’s $G$-function known via Saxena et al. [20], i.e.

$$H_{m_1,n_1}^{m_1,n_1} \left[ x ; q \right] \left( a_1 \right)_1 \left( b_1 \right) = G_{m,n}^{A,B} \left[ x ; q \right] \left( a_1, \cdots, a_A \right) \left( b_1, \cdots, b_B \right).$$
Again, on using the known result, namely

\[
\begin{align*}
\frac{1}{2\pi \omega} \int_C \prod_{j=m_1+1}^{m_1} G(q^{b_j-\ast}) \prod_{j=1}^{n_1} G(q^{1-a_j+\ast}) \frac{\pi x^s}{G(q^s)G(q^{1-s}) \sin \pi s} \, dq_s, \quad (1.14)
\end{align*}
\]

where \(0 \leq m_1 \leq B, 0 \leq n_1 \leq A\) with \(Re[s \log(x) - \log \pi s] < 0\). Also, if we take \(n_1 = 0, m_1 = B\) in the result \(1.14\), we obtain the \(q\)-analog for Mac Robert’s \(E\)-function as:

\[
G_{B,0}^{A,B}[x; q, a_1, \ldots, a_A] = \frac{1}{2\pi \omega} \int_C \prod_{j=m_1+1}^{m_1} G(q^{b_j-\ast}) \prod_{j=1}^{n_1} G(q^{1-a_j+\ast}) \frac{\pi x^s}{G(q^s)G(q^{1-s}) \sin \pi s} \, dq_s, \quad (1.15)
\]

2. Main Results

In this part, we evaluate two theorems which represent \(q\)-Laplace image formulas involving the product of a family of \(q\)-polynomials along with \(q\)-analogue of generalized hypergeometric function.

**Theorem 2.1** Let \(\Re(\lambda) > 0\) and \(\Re[s \log(x) - \log \pi s] < 0\), then the \(q\)-Laplace transform for a product of \(q\)-analogue of \(I\)-function and \(q\)-polynomials family \(f_{n,N}(x; q)\) as the subsequent formula:

\[
qL_p \left\{ x^{\lambda-1} f_{n,N}(x; q) I_{A_i, B_i}^{m_1,n_1} \left( \frac{a_j, \alpha_j, j}{b_j, \beta_j, j} \right) \right\}
\]

\[
= \frac{p^{-\lambda}}{G(q)} \sum_{j=0}^{n/N} \left[ n_{Nj} \right] S_{j,q} p^{-j}
\]

\[
\times I_{A_i+1, B_i}^{m_1, n_1+1} \left( \frac{\rho x^k}{\rho^k} \right) \left( \frac{-j - \lambda, k}{(a_j, \alpha_j, j, (a_j, \alpha_j, j)_{n_1+1, A_i}} \right. \right. \right. 
\]

\[
\left. \left. \left. \left. (b_j, \beta_j, j, (b_j, \beta_j, j)_{m_1+1, B_i} \right) \right) \right), \quad (2.1)
\]

where \(0 \leq m_1 \leq B_i; 0 \leq n_1 \leq A_i; i = 1, 2, \ldots, r; r \text{ is finite; } |q| < 1, \{S_{j,q}\}_{n=0}^{\infty} \text{ be a bounded complex sequence and } \lambda \text{ is arbitrary.}

**Proof:** On making use of results \(1.11\) and \(1.12\), the left hand side (say \(L\)) of the main result \(2.1\) becomes

\[
L = qL_p \left\{ x^{\lambda-1} \sum_{j=0}^{n/N} \left[ n_{Nj} \right] S_{j,q} x^j \frac{1}{2\pi \omega} \int_C \Phi(s) \left( \rho x^k \right)^s \, dq_s \right\},
\]

or

\[
L = \sum_{j=0}^{n/N} \left[ n_{Nj} \right] S_{j,q} x^j \frac{1}{2\pi \omega} \int_C \Phi(s) \rho^s qL_p \left( x^{j+s+k} \right) \, dq_s.
\]

Again, on using the known result, namely

\[
qL_p \left\{ x^{\alpha-1} \right\} = \frac{\Gamma_q(\alpha)(1-q)^{\alpha-1}}{p^\alpha},
\]
the above expression reduce to
\[ L = \sum_{j=0}^{[n/N]} \left[ \frac{n}{N} \right] S_{j,q} \rho^j \frac{1}{2\pi i} \int_C \Phi(s) \rho^s \frac{\Gamma_q(j + \lambda + ks)(1 - q)^{j+\lambda+ks-1}}{p^{j+\lambda+ks}} dq \, ds. \]

On further simplification in above relation, we easily obtain the result (2.1). This completes the proof of theorem.

Now, for \( r = 1, A_1 = A \); and \( B_1 = B \), the result (2.1) yields to the \( q \)-Laplace image formula comprising product of family of \( q \)-polynomials and the \( H_q(.) \) function as folow:

**Theorem 2.2:**- If \( \{S_{n,q}\}_{n=0}^\infty \) be a bounded complex sequence, let \( m_1, n_1; A, B \) be positive integers with that \( 0 \leq m_1 \leq B, 0 \leq n_1 \leq A \) and also \( N \) be a capricious positive integer. Then the following \( q \)-Laplace transform holds:

\[
q L_p \left\{ x^{\lambda-1} f_{n,N}(x, q) H_{A,B}^{m_1,n_1} \left[ x^k; q \begin{array}{c}(a_1,\alpha_1), \ldots, (a_A,\alpha_A) \\ (b_1,\beta_1), \ldots, (b_B,\beta_B)\end{array} \right] \right\}
\]

\[
= \frac{p^\lambda}{G(q)} \sum_{j=0}^{[n/N]} \left[ \frac{n}{N} \right] S_{j,q} p^j \times H_{A+1,B}^{m_1,n_1+1} \left[ s^k; q \begin{array}{c}(-\lambda - j, k), (a_1,\alpha_1), \ldots, (a_A,\alpha_A) \\ (b_1,\beta_1), \ldots, (b_B,\beta_B)\end{array} \right],
\]

provided \( \Re[s \log(x) - \log \sin \pi s] < 0 \) and \( k > 0 \). In same way, certain results are obtained as special cases in the next section.

### 3. Special Cases

In this part, we discuss a few particular cases of the core result, by assigning an appropriate value to the parameters implicated in the results (2.1) and (2.2).

(i). If we put \( N = 1, S_{1,q} = 1, \) if \( j = 0 \) and \( S_{j,q} = 0, \) if \( j \neq 0, \) then \( f_{n,N}(x, q) = 1 \) and we have the following \( q \)-Laplace image formula for the \( q \)-analogue of I-function:

\[
q L_p \left\{ x^{\lambda-1} I_{A,B_i}^{m_1,n_1} \left[ \rho x^k; q \begin{array}{c}(a_j,\alpha_j)_{1,n_i}, (a_{ji},\alpha_{ji})_{n_i+1,A_i} \\ (b_j,\beta_j)_{1,m_i}, (b_{ji},\beta_{ji})_{m_i+1,B_i}\end{array} \right] \right\}
\]

\[
= \frac{1}{p^\lambda} \left\{ I_{A+1,B_i}^{m_1,n_1+1} \left[ \rho \frac{1}{p^\lambda}; q \begin{array}{c}(1 - \lambda, k), (a_j,\alpha_j)_{1,n_i}, (a_{ji},\alpha_{ji})_{n_i+1,A_i} \\ (b_j,\beta_j)_{1,m_i}, (b_{ji},\beta_{ji})_{m_i+1,B_i}\end{array} \right] \right\}, \quad (k > 0),
\]

where \( 0 \leq m_1 \leq B_i; 0 \leq n_1 \leq A_i; \) \( i = 1, 2, \cdots, r; \) \( r \) is finite, \( |q| < 1 \) and for arbitrary \( \lambda \) and \( \rho \).

(ii). Again, for \( r = 1, A_1 = A; \) and \( B_1 = B, \) the result (3.1) yields to the \( q \)-Laplace image formula comprising the \( H_q(.) \) function as

\[
q L_p \left\{ x^{\lambda} H_{A,B}^{m_1,n_1} \left[ \rho x^k; q \begin{array}{c}(a,\alpha) \\ (b,\beta)\end{array} \right] \right\}
\]
\[
\frac{\{G(q)\}^{-1}}{p^{\lambda+1}} H_{A+1,B}^{m_1,n_1+1} \left[ \frac{\rho}{\rho^k + q} \bigg| \begin{array}{c}
(-\lambda, k), (a, \alpha) \\
(b, \beta)
\end{array} \right],
\]
where \(0 \leq m_1 \leq B; 0 \leq n_1 \leq A; |q| < 1\) and for arbitrary \(\lambda\) and \(\rho\).

(iii). Further, if we put \(\rho = 1\) and \(k = 1\), the result \(3.2\) yields to the outcome due to Yadav and Purohit [22], namely
\[
qL_p \left\{ x^\lambda H_{A,B}^{m_1,n_1} \left[ \begin{array}{c}
x; q \\
(a, \alpha) \\
(b, \beta)
\end{array} \right] \right\} = \frac{\{G(q)\}^{-1}}{p^{\lambda+1}} H_{A+1,B}^{m_1,n_1+1} \left[ \frac{1}{p^k + q} \bigg| \begin{array}{c}
(0, 1), (a, \alpha) \\
(b, \beta)
\end{array} \right],
\]
where \(0 \leq m_1 \leq B; 0 \leq n_1 \leq A; |q| < 1\) and for arbitrary \(\lambda\).

(iv). Lastly, it is very intriguing to see that in sight of the point of limit formulae
\[
\lim_{q \to 1^-} (q^n - q) = (\alpha)_n \quad \text{and} \quad \lim_{q \to 1^-} \Gamma_q(\alpha) = \Gamma(\alpha),
\]
the result \(3.3\) is a \(q\)-extension of the identified consequence due to Srivastava et al. [22], namely
\[
qL_p \left\{ x^\lambda H_{A,B}^{m_1,n_1} \left[ \begin{array}{c}
\rho x^k \\
(a, \alpha) \\
(b, \beta)
\end{array} \right] \right\} = \frac{1}{p^{\lambda+1}} H_{A+1,B}^{m_1,n_1+1} \left[ \frac{\rho}{\rho^k + q} \bigg| \begin{array}{c}
(-\lambda, k), (a, \alpha) \\
(b, \beta)
\end{array} \right].
\]

4. Conclusion

We conclude with the remark that the outcomes examined in this paper are common in character and the latest input to the theory of \(q\)-series. It is also significant to mention that the consequences proved in this paper may use to find some solutions of certain \(q\)-integral and \(q\)-difference equations and related to \(q\)-polynomials, the basic analogues of Meijer’s \(G\)-function, Fox’s \(H\)-function and \(I\)-function. Further, one can obtain a number of image formulas involving orthogonal \(q\)-polynomials as particular cases of our results, by giving special values to the sequence \(S_{j,q}\), in the family of polynomials which include the polynomials viz. \(q\)-Laguerre, \(q\)-Hermite, \(q\)-Jacobi, \(q\)-Konhauser polynomials and many others.

References

q-LAPLACE TRANSFORM FOR q-ANALOGUE OF I- FUNCTION


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