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PETROVIĆ-TYPE INEQUALITIES FOR HARMONIC CONVEX FUNCTIONS ON COORDINATES

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Abstract. Harmonic convex functions constitute a very important new class of nonconvex functions. Harmonic convexity has now gained remarkable prominence in the theory of inequalities and applications as well as in other branches of mathematics. Furthermore, harmonic convexity provides an analytic tool in order to estimate several known definite integrals. In this article, we extend Petrović-type inequalities and also establish majorization-type inequalities for harmonic convex functions on coordinates.

1. Introduction and Auxiliary Results

Convexity of functions has been frequently used in various fields of pure and applied mathematics, for instance, in function theory, mathematical analysis, functional analysis, probability theory, optimization theory, operational research, and information theory. In short, convex functions entail a strong and elegant interaction between analysis and geometry. The simple generalization to a convex function extensively widens our scope for analysis. Inequalities are frequently used in solving several problems of the applied sciences. Some recent work on the applications of mathematical inequalities can be found in [2–4, 6, 8, 10–12, 14, 15, 18, 19]. In [20] (see also [21, page 154]), M. Petrović proved the following result, which is known in the literature as Petrović's inequality.

Theorem 1.1. *Suppose $(x_1, \dots, x_n)$ and $(p_1, \dots, p_n)$ are nonnegative n-tuples such that $\sum_{k=1}^n p_k x_k \geq x_i$ for $i = 1, \dots, n$ and $x_i, \sum_{k=1}^n p_k x_k \in [0, a]$. If $f : [0, a] \to \mathbb{R}$ is convex, then the inequality*

$$\sum_{k=1}^{n} p_k f(x_k) \leq f\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{k=1}^{n} p_k - 1\right) f(0) \tag{1.1}$$

holds.

Definition 1.2. Let $n \geq 2$, $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$ be n-tuples of real numbers and

$$a_{[1]} \geq a_{[2]} \geq \dots \geq a_{[n]}, \quad b_{[1]} \geq b_{[2]} \geq \dots \geq b_{[n]}$$

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be their ordered arrangements. Then we say that $\mathbf{a}$ *majorizes* $\mathbf{b}$ (in symbols $\mathbf{a} \succ \mathbf{b}$) if

$$\sum_{i=1}^{k} a_{[i]} \geq \sum_{i=1}^{k} b_{[i]}, \quad k = 1, \ldots, n-1 \quad \text{and} \quad \sum_{i=1}^{n} a_{[i]} = \sum_{i=1}^{n} b_{[i]}.$$

During the investigation of convexity, many researchers established new classes of functions that are not convex in general. Some of them are the so-called harmonic convex functions [16], harmonic (α, m)-convex functions [17], harmonic (s, m)-convex functions [7,9], and harmonic $(p,(s,m))$-convex functions [10]. For a quick glance on the importance of these classes and applications, see [5] and references therein. Here we recall these definitions for better understanding of our work.

Definition 1.3. A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be *harmonic convex* on I if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.2}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the inequality is reversed, then f is said to be *harmonic concave*.

Definition 1.4 (See [17])**.** A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be *harmonic (α, m)-convex* on I if

$$f\left(\frac{mxy}{mty + (1-t)x}\right) \leq t^{\alpha} f(x) + m(1 - t^{\alpha}) f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$, where $\alpha \in [0,1]$ and $m \in (0,1]$. If the inequality is reversed, then f is said to be *harmonic (α, m)-concave*.

Definition 1.5 (See [7,9])**.** A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be *harmonic (s, m)-convex* on I if

$$f\left(\frac{mxy}{mty + (1-t)x}\right) \leq t^{s} f(x) + m(1-t)^{s} f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$, where $s \in (0,1]$ and $m \in (0,1]$. If the inequality is reversed, then f is said to be *harmonic (s, m)-concave*.

Definition 1.6 (See [10])**.** A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be *harmonic $(p,(s,m))$-convex* on I if

$$f\left(\frac{mxy}{[t(my)^p + (1-t)x^p]^{\frac{1}{p}}}\right) \leq t^{s} f(x) + m(1-t)^{s} f(y)$$

holds for all $x, y \in I$ with $my \in I$ and $t \in [0,1]$, where $p \in \mathbb{R} \setminus \{0\}$, $s, m \in (0,1]$. If the inequality is reversed, then f is said to be *harmonic $(p,(s,m))$-concave*.

In [5], Baloch et al. established the following relations between classes of convex and harmonic convex functions.

Lemma 1.7 (See [5])**.** *Let $I \subseteq \mathbb{R} \setminus \{0\}$ be an interval. Put*

$$I^{-1} := \left\{ y \in \overline{\mathbb{R}} : \ y = \frac{1}{x}, \ x \in I \right\}.$$

A function $f : I \to \mathbb{R}$ is harmonic convex if and only if $g : I^{-1} \to \mathbb{R}$ is convex, where g is defined as $g(y) = f(1/y)$.

Lemma 1.8 (See [5]). *Let $I \subseteq (0,\infty)$. A function $f : I \to \mathbb{R}$ is harmonic convex if and only if $g : I \to \mathbb{R}$ is convex, where g is defined as $g(x) = xf(x)$.*

Example 1.9 (See [5]).

1. Let $f : (0,\infty) \to \mathbb{R}$ and $g : (-\infty, 0) \to \mathbb{R}$ be defined by $f(x) = g(x) = x$. Then f is harmonic convex and g is harmonic concave.
2. Let $f : (0,\infty) \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} x & \text{if} \quad 0 < x < 2, \\ 4 - \frac{4}{x} & \text{if} \quad x \geq 2. \end{cases}$$
Then f is harmonic convex on $(0,\infty)$, since $xf(x)$ is convex on $(0,\infty)$.
3. Let $f : (0,\infty) \to \mathbb{R}$ be defined by
$$f(x) = \frac{(x-1)^2 + 1}{x}.$$
Then f is harmonic convex on $(0,\infty)$, since $xf(x)$ is convex on $(0,\infty)$.

In [1], Baloch and Chu proved the following Petrović-type and majorization-type inequalities.

Theorem 1.10. *Suppose $(x_1, \dots, x_n)$ and $(p_1, \dots, p_n)$ are positive n-tuples such that $\sum_{k=1}^n p_k x_k \geq x_i > c$ for $i = 1, \dots, n$ and $c, x_i, \sum_{k=1}^n p_k x_k \in [\varepsilon, a]$. If the function $f : [\varepsilon, a] \to \mathbb{R}$ is harmonic convex, then*

$$\begin{aligned} \sum_{k=1}^n p_k x_k f(x_k) \leq & \frac{\sum_{k=1}^n p_k (x_k - c)}{\sum_{k=1}^n p_k x_k - c} \left(\sum_{k=1}^n p_k x_k \right) f \left(\sum_{k=1}^n p_k x_k \right) \\ & + c \left(\sum_{k=1}^n p_k - \frac{\sum_{k=1}^n p_k (x_k - c)}{\sum_{k=1}^n p_k x_k - c} \right) f(c). \end{aligned} \tag{1.3}$$

Theorem 1.11. *Let $\boldsymbol{a} = (a_1, \dots, a_n)$, $\boldsymbol{b} = (b_1, \dots, b_n)$ be finite sequences from $I \subseteq \mathbb{R} \setminus \{0\}$, $\boldsymbol{p} = (p_1, \dots, p_n)$ be a positive n-tuple. If $\boldsymbol{a} \succ \boldsymbol{b}$ and $f : I \to \mathbb{R}$ is harmonic convex, then the inequality*

$$\sum_{i=1}^n p_i a_i f(a_i) \geq \sum_{i=1}^n p_i b_i f(b_i) \tag{1.4}$$

holds.

Now, we give the definitions of harmonic convex and coordinated harmonic convex functions on rectangles, and then establish some basic results for them, which will be used in the main section of this article.

Definition 1.12. Consider a rectangle $\Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \setminus \{(0,0)\}$ with $a < b$ and $c < d$. A function $f : \Delta \to \mathbb{R}$ is said to be *harmonic convex* on Δ if

$$f \left(\frac{xy}{tx + (1-t)y}, \frac{uv}{tu + (1-t)v} \right) \leq tf(y,v) + (1-t)f(x,u)$$

for all $x, y \in [a,b]$, $u, v \in [c,d]$, $t \in [0,1]$.

Definition 1.13. Consider a rectangle $\Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \setminus \{(0,0)\}$ with $a < b$ and $c < d$. A function $f : \Delta \to \mathbb{R}$ is said to be *coordinated harmonic convex* on Δ if

$$\begin{aligned} f \left(\frac{xy}{tx + (1-t)y}, \frac{uv}{\lambda u + (1-\lambda)v} \right) \leq & (1-t)(1-\lambda)f(x,u) + (1-t)\lambda f(x,v) \\ & + t(1-\lambda)f(y,u) + t\lambda f(y,v) \end{aligned}$$

for all $x, y \in [a, b]$, $u, v \in [c, d]$, $t, \lambda \in [0, 1]$.

Remark 1.14. Here, we would like to point out that a function $f : \Delta \to \mathbb{R}$ is coordinated harmonic convex on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}$ and $f_x : [c, d] \to \mathbb{R}$ defined by $f_y(u) := f(u, y)$ and $f_x(v) := f(x, v)$ are harmonic convex for all $x \in [a, b]$ and $y \in [c, d]$.

Lemma 1.15. *Let $\Delta = [a, b] \times [c, d] \to \mathbb{R}^2 \setminus \{(0, 0)\}$. Put*

$$\Delta^{-1} := \left\{ (u, v) \in \overline{\mathbb{R}}^2 : \ u = \frac{1}{x},\ v = \frac{1}{y},\ (x, y) \in \Delta \right\}.$$

A function $f : \Delta \to \mathbb{R}$ is harmonic convex (coordinated harmonic convex) on Δ if and only if $g : \Delta^{-1} \to \mathbb{R}$ is convex (coordinated convex) on Δ, where g is defined as $g(u, v) = f(1/u, 1/v)$.

Proof. The idea of proof is similar to the proof of [5, Lemma 1.1], and we omit the calculations. □

Lemma 1.16. *Let $\Delta = [a, b] \times [c, d] \to \mathbb{R}^2 \setminus \{(0, 0)\}$. A function $f : \Delta \to \mathbb{R}$ is coordinated harmonic convex on Δ if and only if $g_x, g_y : \Delta \to \mathbb{R}$ are convex on Δ, where g_x, g_y are defined by $g_x(y) = y f_x(y)$ and $g_y(x) = x f_y(x)$.*

Proof. The idea of proof is similar to the proof of [5, Lemma 1.2], and we omit the calculations. □

Lemma 1.17. *Every harmonic convex function $f : \Delta \to \mathbb{R}$ is coordinated harmonic convex on Δ, but the converse does not hold in general.*

Proof. Suppose that $f : \Delta \to \mathbb{R}$ is harmonic convex on Δ. Consider $f_x : [c, d] \to \mathbb{R}$ defined by $f_x(u) := f(x, u)$. Then, for all $\lambda \in [0, 1]$ and $u, v \in [c, d]$, we have

$$\begin{aligned} f_x\left(\frac{uv}{\lambda v + (1-\lambda)u}\right) &= f\left(x, \frac{uv}{\lambda v + (1-\lambda)u}\right) \\ &= f\left(\frac{x^2}{\lambda x + (1-\lambda)x}, \frac{uv}{\lambda v + (1-\lambda)u}\right) \\ &\leq \lambda f(x, u) + (1-\lambda) f(x, v) \\ &= \lambda f_x(u) + (1-\lambda) f_x(v), \end{aligned}$$

which shows harmonic convexity of f_x. The fact that $f_y : [a, b] \to \mathbb{R}$ defined by $f_y(v) := f(v, y)$ is also harmonic convex for all $y \in [c, d]$ goes likewise, and we shall omit the details. Now, consider the mapping

$$f_\infty : \Delta^{-1} = [1, \infty]^2 \to [0, 1] \quad \text{defined by} \quad f_\infty(x, y) := \frac{1}{xy}.$$

We have

$$g_\infty(u, v) := f_\infty\left(\frac{1}{u}, \frac{1}{v}\right) = uv,$$

which is coordinated convex but not convex on $\Delta = [0, 1]^2$ (see [13] for details). Thus, by Lemma 1.15, f_∞ is coordinated harmonic convex on Δ^{-1} but not harmonic convex on Δ^{-1}. □

2. Main Results

Theorem 2.1. *Let $\Delta = [\varepsilon, a] \times [\varepsilon, b]$, $(x_1, \ldots, x_n) \in [\varepsilon, a]^n$, $(y_1, \ldots, y_n) \in [\varepsilon, b]^n$, $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ be positive n-tuples such that*

$$\sum_{k=1}^{n} p_k x_k \geq x_i > c \quad \text{and} \quad \sum_{k=1}^{n} q_k y_k \geq y_i > c \quad \text{for} \quad i = 1, \ldots, n.$$

Also, let $c, x_i, \sum_{k=1}^{n} p_k x_k \in [\varepsilon, a]$ and $c, y_i, \sum_{k=1}^{n} q_k y_k \in [\varepsilon, b]$. If $f : \Delta \to \mathbb{R}$ is harmonic convex on Δ, then

$$\begin{aligned}\sum_{k=1}^{n}\sum_{\ell=1}^{n} p_k q_\ell x_k y_\ell f(x_k, y_\ell) \leq & A_1(\boldsymbol{x}, \boldsymbol{p}, c) B_1(\boldsymbol{y}, \boldsymbol{q}, c) f\left(\sum_{k=1}^{n} p_k x_k, \sum_{\ell=1}^{n} q_\ell y_\ell\right) \\ & + A_1(\boldsymbol{x}, \boldsymbol{p}, c) B_2(\boldsymbol{y}, \boldsymbol{q}, c) f\left(\sum_{k=1}^{n} p_k x_k, c\right) \\ & + A_2(\boldsymbol{x}, \boldsymbol{p}, c) B_1(\boldsymbol{y}, \boldsymbol{q}, c) f\left(c, \sum_{\ell=1}^{n} q_\ell y_\ell\right) \\ & + A_2(\boldsymbol{x}, \boldsymbol{p}, c) B_2(\boldsymbol{y}, \boldsymbol{q}, c) f(c, c),\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}A_1(\boldsymbol{x}, \boldsymbol{p}, c) &:= \frac{\sum_{k=1}^{n} p_k (x_k - c)}{\sum_{k=1}^{n} p_k x_k - c}\left(\sum_{k=1}^{n} p_k x_k\right), \\ A_2(\boldsymbol{x}, \boldsymbol{p}, c) &:= c\left(\sum_{k=1}^{n} p_k - \frac{\sum_{k=1}^{n} p_k (x_k - c)}{\sum_{k=1}^{n} p_k x_k - c}\right), \\ B_1(\boldsymbol{y}, \boldsymbol{q}, c) &:= \frac{\sum_{\ell=1}^{n} q_\ell (y_\ell - c)}{\sum_{\ell=1}^{n} q_\ell y_\ell - c}\left(\sum_{\ell=1}^{n} q_\ell y_\ell\right), \\ B_2(\boldsymbol{y}, \boldsymbol{q}, c) &:= c\left(\sum_{\ell=1}^{n} q_\ell - \frac{\sum_{\ell=1}^{n} q_\ell (y_\ell - c)}{\sum_{\ell=1}^{n} q_\ell y_\ell - c}\right).\end{aligned}$$

Proof. Let $f_x : [\varepsilon, b] \to \mathbb{R}$ and $f_y : [\varepsilon, a] \to \mathbb{R}$ be mappings such that $f_x(y) = f(x, y)$ and $f_y(x) = f(x, y)$. Since f is coordinated harmonic convex on Δ, f_y is harmonic convex on $[\varepsilon, a]$. By Theorem 1.10, one has

$$\begin{aligned}\sum_{k=1}^{n} p_k x_k f_y(x_k) \leq & \frac{\sum_{k=1}^{n} p_k (x_k - c)}{\sum_{k=1}^{n} p_k x_k - c}\left(\sum_{k=1}^{n} p_k x_k\right) f_y\left(\sum_{k=1}^{n} p_k x_k\right) \\ & + c\left(\sum_{k=1}^{n} p_k - \frac{\sum_{k=1}^{n} p_k (x_k - c)}{\sum_{k=1}^{n} p_k x_k - c}\right) f_y(c).\end{aligned}$$

By assuming $y = y_\ell$, we have

$$\begin{aligned}\sum_{k=1}^{n} p_k x_k f(x_k, y_\ell) \leq & \frac{\sum_{k=1}^{n} p_k (x_k - c)}{\sum_{k=1}^{n} p_k x_k - c}\left(\sum_{k=1}^{n} p_k x_k\right) f\left(\sum_{k=1}^{n} p_k x_k, y_\ell\right) \\ & + c\left(\sum_{k=1}^{n} p_k - \frac{\sum_{k=1}^{n} p_k (x_k - c)}{\sum_{k=1}^{n} p_k x_k - c}\right) f(c, y_\ell).\end{aligned}$$

This gives

$$
\begin{aligned}
&\sum_{k=1}^{n}\sum_{\ell=1}^{n} p_k q_\ell x_k y_\ell f(x_k, y_\ell) \\
&\leq \frac{\sum_{k=1}^{n} p_k(x_k - c)}{\sum_{k=1}^{n} p_k x_k - c}\left(\sum_{k=1}^{n} p_k x_k\right)\sum_{\ell=1}^{n} q_\ell y_\ell f\left(\sum_{k=1}^{n} p_k x_k, y_\ell\right) \\
&+ c\left(\sum_{k=1}^{n} p_k - \frac{\sum_{k=1}^{n} p_k(x_k - c)}{\sum_{k=1}^{n} p_k x_k - c}\right)\sum_{\ell=1}^{n} q_\ell y_\ell f(c, y_\ell).
\end{aligned} \tag{2.2}
$$

Again using Theorem 1.10 on the terms of the right-hand side for the second coordinates, we have

$$
\begin{aligned}
&\sum_{\ell=1}^{n} q_\ell y_\ell f\left(\sum_{k=1}^{n} p_k x_k, y_\ell\right) \\
&\leq \frac{\sum_{\ell=1}^{n} q_\ell(y_\ell - c)}{\sum_{\ell=1}^{n} q_\ell y_\ell - c}\left(\sum_{\ell=1}^{n} q_\ell y_\ell\right) f\left(\sum_{k=1}^{n} p_k x_k, \sum_{\ell=1}^{n} q_\ell y_\ell\right) \\
&+ c\left(\sum_{\ell=1}^{n} q_\ell - \frac{\sum_{\ell=1}^{n} q_\ell(y_\ell - c)}{\sum_{\ell=1}^{n} q_\ell y_\ell - c}\right) f\left(\sum_{k=1}^{n} p_k x_k, c\right)
\end{aligned} \tag{2.3}
$$

and

$$
\begin{aligned}
\sum_{\ell=1}^{n} q_\ell y_\ell f(c, y_\ell) \leq &\frac{\sum_{\ell=1}^{n} q_\ell(y_\ell - c)}{\sum_{\ell=1}^{n} q_\ell y_\ell - c}\left(\sum_{\ell=1}^{n} q_\ell y_\ell\right) f\left(c, \sum_{\ell=1}^{n} q_\ell y_\ell\right) \\
&+ c\left(\sum_{\ell=1}^{n} q_\ell - \frac{\sum_{\ell=1}^{n} q_\ell(y_\ell - c)}{\sum_{\ell=1}^{n} q_\ell y_\ell - c}\right) f(c, c).
\end{aligned} \tag{2.4}
$$

Using inequalities (2.3) and (2.4) in inequality (2.2), we get inequality (2.1). □

Remark 2.2. If f is strictly coordinated harmonic convex, then inequality (2.1) is strict unless all x_k and y_ℓ are not equal or $\sum_{k=1}^{n} p_k \neq 1$ and $\sum_{\ell=1}^{n} q_\ell \neq 1$.

Remark 2.3. If we assume $y_\ell = c$ and $q_\ell = 1$ for $\ell = 1, \ldots, n$ with $f(x_k, c) = f(x_k)$ and $f(c, c) = f(c)$, then we get inequality (1.3).

Theorem 2.4. *Let $\Delta = (0, \infty)^2$, $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n)$ be n-tuples and $\boldsymbol{c} = (c_1, \ldots, c_m)$, $\boldsymbol{d} = (d_1, \ldots, d_m)$ be m-tuples such that $a_i, b_i, c_j, d_j \in (0, \infty)$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. If $\boldsymbol{a} \succ \boldsymbol{b}$ and $\boldsymbol{c} \succ \boldsymbol{d}$, then the inequality*

$$
\sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j a_i c_j f(a_i, c_j) \geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j b_i d_j f(b_i, d_j) \tag{2.5}
$$

holds for each harmonic convex $f : \Delta \to \mathbb{R}$.

Proof. Let $f_x, f_y : (0, \infty) \to \mathbb{R}$ be mappings such that $f_x(y) = f_y(x) = f(x, y)$. Since f is harmonic convex on Δ, f_y is harmonic convex on $(0, \infty)$. By Theorem 1.11, one has

$$
\sum_{i=1}^{n} p_i a_i f(a_i, y) \geq \sum_{i=1}^{n} p_i b_i f(b_i, y). \tag{2.6}
$$

Again, by using Theorem 1.11 for harmonic convexity of f_x on $(0,\infty)$, we get

$$\sum_{j=1}^{m} q_j c_j f(a_i, c_j) \geq \sum_{j=1}^{m} q_j d_j f(a_i, d_j),$$

which gives

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j a_i c_j f(a_i, c_j) \geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j a_i d_j f(a_i, d_j). \tag{2.7}$$

Now, if we assume that $y = d_j$ in inequality (2.6), then we get

$$\sum_{i=1}^{n} p_i a_i f(a_i, d_j) \geq \sum_{i=1}^{n} p_i b_i f(b_i, d_j),$$

which gives

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j a_i d_j f(a_i, d_j) \geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j b_i d_j f(b_i, d_j). \tag{2.8}$$

By combing inequalities (2.7) and (2.8), we get the inequality (2.5). □

3. Conclusion

In this article, we extended Petrović's inequality to rectangles for harmonic convex functions by using certain conditions. The methods and techniques used in this article are new.

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Conflict of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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