Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 11 Issue 1(2020), Pages 43-63.

FINDING MIXED FAMILIES OF SPECIAL POLYNOMIALS ASSOCIATED WITH GOULD-HOPPER MATRIX POLYNOMIALS

GHAZALA YASMIN, HIBAH ISLAHI, SERKAN ARACI

Abstract. In this article, the general polynomials are taken as base with the Gould-Hopper matrix polynomials to introduce a family of 3-variable general-Gould-Hopper matrix polynomials (3VgGHMaP). These polynomials are framed within the context of monomiality principle and their properties are established. Examples of some members belonging to this family are considered. Certain bilateral and bilinear generating matrix functions for 3Vg-GHMaP are also derived.

1. introduction and preliminaries

Special functions of more than one variable have diverse applications in physics and engineering. The special polynomials of two variables are important from the point of view of applications and also these polynomials are helpful in introducing new families of special polynomials. We recall that the general class of 2-variable polynomials, namely 2-variable general polynomials (2VgP) $p_n(x,y)$ is considered in [12]. These polynomials are defined by the generating function [12, p.4(14)]

$$e^{xt}\Phi(y,t)=\sum_{n=0}^{\infty}p_n(x,y)\frac{t^n}{n!},\quad p_0(x,y)=1, \tag{1.1}$$

where $\Phi(y,t)$ has (at least the formal) series expansion

$$\Phi(y,t)=\sum_{n=0}^{\infty}\phi_n(y)\frac{t^n}{n!},\quad \phi_0(y)\neq 0. \tag{1.2}$$

The 2VgP family contains very important polynomials such as the Gould-Hopper polynomials (GHP) $H_n^{(s)}(x,y)$, 2-variable Hermite Kampé de Feriet polynomials (2VHKdFP) $H_n(x,y)$, 2-variable generalized Laguerre polynomials (2VGLP) ${}_sL_n(y,x)$, 2-variable Laguerre polynomials (2VLP) $L_n(y,x)$, 2-variable truncated exponential polynomials (2VTEP) (of order r) $e_n^{(r)}(x,y)$, two-dimensional Bernoulli polynomials (2DBP) $B_n^{(j)}(x,y)$ and two-dimensional Euler Polynomials (2DEP) $E_n^{(j)}(x,y)$. We present the list of some known 2VgP family in Table 1.

2000 *Mathematics Subject Classification.* 15A60, 33C45, 33C47.

Key words and phrases. 2-variable general polynomials; Gould-Hopper matrix polynomials; Monomiality principle; Operational techniques.

Submitted December 11, 2019. Published February 19, 2020.
Communicated by Toufik Mansour.

TABLE 1. Certain members belonging to the 2VgP $p_n(x,y)$.

S.No.	$\mathbf{\Phi(y,t)}$	**Generating Functions**	**Polynomials**
I.	e^{yt^s}	$\exp(xt+yt^s)=\sum_{n=0}^{\infty} H_n^{(s)}(x,y)\frac{t^n}{n!}$	Gould-Hopper polynomials (GHP) [9]
II.	e^{yt^2}	$\exp(xt+yt^2)=\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}$	2-variable Hermite Kampé de Feriet polynomials (2VHKdFP) [1]
III.	$C_0(-yt^s)$	$\exp(xt)C_0(-yt^s)=\sum_{n=0}^{\infty} {}_sL_n(y,x)\frac{t^n}{n!}$	2-variable generalized Laguerre polynomials (2VGLP) [6]
IV.	$C_0(yt)$	$\exp(xt)C_0(yt)=\sum_{n=0}^{\infty} L_n(y,x)\frac{t^n}{n!}$	2-variable Laguerre polynomials (2VLP) [5]
V.	$\frac{1}{1-yt^r}$	$\frac{e^{xt}}{1-yt^r}=\sum_{n=0}^{\infty} e_n^{(r)}(x,y)\frac{t^n}{n!}$	2-variable truncated exponential polynomials (2VTEP) (of order r) [7]
VI.	$\frac{te^{yt^j}}{e^t-1}$	$\frac{t}{e^t-1}e^{xt+yt^j}=\sum_{n=0}^{\infty} B_n^{(j)}(x,y)\frac{t^n}{n!}$; $\|t\|<2\pi$	2-dimensional Bernoulli polynomials (2DBP) [3]
VII.	$\frac{2e^{yt^j}}{e^t+1}$	$\frac{2}{e^t+1}e^{xt+yt^j}=\sum_{n=0}^{\infty} E_n^{(j)}(x,y)\frac{t^n}{n!}$; $\|t\|<\pi$	2-dimensional Euler Polynomials (2DEP) [18]

It is shown in [12], that the polynomials $p_n(x,y)$ are quasi-monomial [5, 16] with respect to the following multiplicative and derivative operators:

$$\hat{M}_p = x + \frac{\Phi'(y,\partial_x)}{\Phi(y,\partial_x)} \quad \left(\partial_x := \frac{\partial}{\partial x} \text{ and } \Phi'(y,t) := \frac{\partial}{\partial t}\Phi(y,t)\right) \tag{1.3}$$

and

$$\hat{P}_p = \partial_x, \tag{1.4}$$

respectively.

According to the monomiality principle and in view of equations (1.3) and (1.4), we have

$$\hat{M}_p\{p_n(x,y)\} = p_{n+1}(x,y) \tag{1.5}$$

and

$$\hat{P}_p\{p_n(x,y)\} = np_{n-1}(x,y), \tag{1.6}$$

respectively.

Now, since the 2VgP $p_n(x,y)$ are quasi-monomial, the properties of these polynomials can be derived from those of the multiplicative and derivative operators $\hat{M}_p$ and $\hat{P}_p$ respectively. In fact, we have

$$\hat{M}_p\hat{P}_p\{p_n(x,y)\} = n\ p_n(x,y), \tag{1.7}$$

which yields the following differential equation satisfied by $p_n(x,y)$:

$$\left(x\partial_x + \frac{\Phi'(y,\partial_x)}{\Phi(y,\partial_x)}\partial_x - n\right)p_n(x,y) = 0. \tag{1.8}$$

Again since $p_0(x,y)=1$, the 2VgP $p_n(x,y)$ can be explicitly constructed as:

$$p_n(x,y) = \hat{M}_p^n\{p_0(x,y)\} = \hat{M}_p^n\{1\}. \tag{1.9}$$

Identity (1.9) implies that the exponential generating function of the 2VgP $p_n(x,y)$ can be cast in the form :

$$\exp(\hat{M}_p t)\{1\} = \sum_{n=0}^{\infty} p_n(x,y)\frac{t^n}{n!}, \quad |t| < \infty, \tag{1.10}$$

which yields generating function (1.1). Again, expanding the exponential function e^{xt} and using series expansion (1.2) in the left hand side (l.h.s.) of generating function (1.1), we get the following series definition of the 2VgP $p_n(x,y)$:

$$p_n(x,y) = n!\sum_{k=0}^{n} \frac{x^{n-k}\phi_k(y)}{(n-k)!k!}. \tag{1.11}$$

It can be easily verified that the $\hat{M}_p$ and $\hat{P}_p$ satisfy the following commutation relation:

$$[\hat{P}_p, \hat{M}_p] = 1, \tag{1.12}$$

which deals with combinatorics of commutation relations [13].

An extension to the matrix framework of the classical special polynomials have been extensively studied and investigated in recent years [2, 4, 10, 14, 17]. Matrix polynomials are important due to their applications in certain areas of statistics, physics and engineering and is an emergent field.

Throughout the paper unless otherwise stated, we assume that A is a positive stable matrix in $\mathbb{C}^{N\times N}$, that is, A satisfies following condition:

$$Re(\mu) > 0, \quad \text{for all } \mu \in \sigma(A), \tag{1.13}$$

where $\sigma(A)$ denotes the set of all the eigenvalues of A.

If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principal logarithm of z, then $z^{\frac{1}{2}}$ represents $\exp\left(\frac{1}{2}\log(z)\right)$. If A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset D_0$, then $A^{\frac{1}{2}} = \sqrt{A}$ denotes the image by $z^{\frac{1}{2}}$ of the matrix functional calculus [8] acting on the matrix A.

We consider Gould-Hopper matrix polynomials (GHMaP) $g_n^m(x,y;A,B)$ [4], which are defined by the series [4, p.81(2.2)]

$$g_n^m(x,y;A,B) = n!\sum_{k=0}^{[\frac{n}{m}]} \frac{(\sqrt{2A})^{n-mk}(B)^k}{(n-mk)!k!} x^{n-mk}y^k \tag{1.14}$$

and specified by the generating function [4, p.81(2.1)]

$$\exp(x\sqrt{2A}\ t)\ \exp(Byt^m) = \sum_{n=0}^{\infty} g_n^m(x,y;A,B)\frac{t^n}{n!}, \tag{1.15}$$

where A and B are matrices in $\mathbb{C}^{N\times N}$ and m is a positive integer.

In view of monomiality principle (1.5) and (1.6) (for GHMaP $g_n^m(x,y;A,B)$) and equation (1.15), we find that the GHMaP $g_n^m(x,y;A,B)$ are quasi-monomial [5, 16] under the action of the operators

$$\hat{M}_G = x\sqrt{2A} + my(\sqrt{2A})^{-(m-1)}B\partial_x^{(m-1)} \tag{1.16}$$

and

$$\hat{P}_G = \frac{1}{\sqrt{2A}}\partial_x, \tag{1.17}$$

respectively.

Now, since the GHMaP $g_n^m(x,y;A,B)$ are quasi-monomial, the properties of these polynomials can be derived from monomiality principle. In view of the monomiality principle (1.7) (for GHMaP $g_n^m(x,y;A,B)$), the differential equation satisfied by GHMaP $g_n^m(x,y;A,B)$ is given by

$$\Big(my(\sqrt{2A})^{-m}B\partial_x^m + x\partial_x - n\Big)\, g_n^m(x,y;A,B) = 0. \tag{1.18}$$

From monomiality principle (1.9) (for GHMaP $g_n^m(x,y;A,B)$), the GHMaP $g_n^m(x,y;A,B)$ can be explicitly constructed as

$$g_n^m(x,y;A,B) = \Big(x\sqrt{2A} + my(\sqrt{2A})^{-(m-1)}B\partial_x^{m-1}\Big)^n \{1\}, \tag{1.19}$$

which yields series definition (1.14) of GHMaP $g_n^m(x,y;A,B)$.

Monomiality principle (1.10) (for GHMaP $g_n^m(x,y;A,B)$) implies that the exponential generating function of the GHMaP $g_n^m(x,y;A,B)$ can be cast in the form

$$\exp(\hat{M}_G t)\{1\} = \sum_{n=0}^{\infty} g_n^m(x,y;A,B)\frac{t^n}{n!}, \tag{1.20}$$

which on using the relation

$$\partial_y g_n^m(x,y;A,B) = (\sqrt{2}A)^{(-m)}\ B\ \partial_x^m\ g_n^m(x,y;A,B), \tag{1.21}$$

gives the generating function (1.15).

The formal solution of equation (1.21) along with the initial condition

$$g_n^m(x,0;A,B) = (\sqrt{2A})^n x^n, \tag{1.22}$$

is given by the following operational representation

$$g_n^m(x,y;A,B) = \exp(y(\sqrt{2A})^{-m}B\partial_x^m)\{(x\sqrt{2A})^n\}. \tag{1.23}$$

Taking suitable values of the indices and variables in (1.14) and (1.15), we get certain new and known special polynomials as special cases of the GHMaP $g_n^m(x,y;A,B)$. Some of the known cases are mentioned in Table 2.

Table 2. Certain members belonging to the GHMaP $g_n^m(x,y;A,B)$.

S.No.	Values of the matrices and indices	Relation between the GHMaP $\mathbf{g_n^m(x,y;A,B)}$ and its special cases	Name of the resultant special polynomials
I.	$A \to \frac{r}{2}A, B = I \in \mathbb{C}^{N\times N}$	$g_n^m(x,y;\frac{r}{2}A,I) = H_{n,r}^{(m)}(x,y;A)$	3-index 2-variable Hermite matrix polynomials (3I2VHMaP)[10]
II.	$A \to \frac{m}{2}A, B = -I \in \mathbb{C}^{N\times N}$	$g_n^m(x,y;\frac{m}{2}A,-I) = H_{n,m}(x,y,A)$	Generalized Hermite matrix polynomials (GHMaP)[14]
III.	$A = \frac{1}{2} \in \mathbb{C}^{1\times 1}, B = 1 \in \mathbb{C}^{1\times 1}$	$g_n^m(x,y;\frac{1}{2},1) = H_n^{(m)}(x,y)$	Gould-Hopper polynomials (GHP)[9]
IV.	$B = -I \in \mathbb{C}^{N\times N}, m = 2$	$g_n^2(x,y;A,-I) = H_n(x,y,A)$	2-variable Hermite matrix polynomials (2VHMaP) [2]
V.	$A \to \frac{1}{2} \in \mathbb{C}^{1\times 1}, B = 1 \in \mathbb{C}^{1\times 1}$ $m = 2$	$g_n^2(x,y;\frac{1}{2},1) = H_n(x,y)$	2-variable Hermite Kampé de Feriet polynomials (2VHKdFP)[1]

The operational methods combined with the monomiality principle open new possibilities to deal with the theoretical foundations of special polynomials and also to introduce new families of mixed type special polynomials. Recently the mixed type special matrix polynomials are considered and studied by few authors [11, 17]. The importance of this new class of matrix polynomials have been recognized both in purely mathematical and applied frameworks.

Motivated by the recent work on mixed type polynomials and the importance of operational methods in introducing new families of mixed type special matrix polynomials, a family of 3-variable general-Gould-Hopper matrix polynomials (3VgGHMaP) are introduced in section 2. In section 3, certain examples of some members belonging to this family are considered and their properties are derived. In section 4, bilateral and bilinear relation are established involving these members.

2. General polynomials based Gould-Hopper matrix polynomials

In this section, we introduce the 3-variable general-Gould-Hopper matrix polynomials (3VgGHMaP) denoted by ${}_pg_n^m(x,y,z;A,B)$ by means of generating function and series definition. Further, certain properties of these polynomials are also explored.

First, we derive the generating function for the 3VgGHMaP ${}_pg_n^m(x,y;A,B)$ by proving the following result:

Theorem 2.1. *The generating function for the 3VgGHMaP* ${}_pg_n^m(x,y,z;A,B)$ *is given as:*

$$\exp\left(x\sqrt{2A}\ t\right)\Phi\left(y,\sqrt{2A}\ t\right)\exp\left(Bzt^m\right)=\sum_{n=0}^{\infty}{}_pg_n^m\left(x,y,z;A,B\right)\frac{t^n}{n!}. \tag{2.1}$$

Proof. Replacing x by the multiplicative operator $\hat{M}_p$ of the 2VgP $p_n(x,y)$ and y by z in the left hand side (l.h.s.) and right hand side (r.h.s.) of the generating function (1.15) of GHMaP $g_n^m(x,y;A,B)$, we have

$$\exp\left(\hat{M}_p\sqrt{2A}\ t\right)\exp\left(Bzt^m\right)=\sum_{n=0}^{\infty}g_n^m\left(\hat{M}_p,z;A,B\right)\frac{t^n}{n!}. \tag{2.2}$$

Now using equation (1.10) in the l.h.s. and equation (1.3) in the r.h.s. of equation (2.2), we find

$$\sum_{n=0}^{\infty}p_n(x,y)\frac{(\sqrt{2A}t)^n}{n!}\exp(Bzt^m)=\sum_{n=0}^{\infty}g_n^m\left(x+\frac{\Phi'(y.\partial_x)}{\Phi(y.\partial_x)},z;A,B\right)\frac{t^n}{n!}, \tag{2.3}$$

which on using equation (1.1) in the l.h.s. and denoting the resultant in the r.h.s. by ${}_pg_n^m(x,y,z;A,B)$, that is

$$g_n^m(\hat{M}_p,z;A,B)=g_n^m\left(x+\frac{\Phi'(y,\partial_x)}{\Phi(y,\partial_x)},z;A,B\right)={}_pg_n^m(x,y,z;A,B), \tag{2.4}$$

yields assertion (2.1). □

Remark. We remark that, equation (2.4) gives the operational correspondence between the GHMaP $g_n^m(x,z;A,B)$ and 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$.

Next, we obtain the series definition of the 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$ by proving the following result:

Theorem 2.2. *The 3VgGHMaP* ${}_pg_n^m(x,y,z;A,B)$ *are defined by the series:*

$$ {}_pg_n^m(x,y,z;A,B) = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(\sqrt{2A})^{n-mk}B^kz^k}{(n-mk)!k!}p_{n-mk}(x,y) \tag{2.5}$$

and

$$ {}_pg_n^m(x,y,z;A,B) = n!\sum_{k=0}^{n}\frac{(\sqrt{2A})^{k}\phi_k(y)}{(n-k)!k!}g_{n-k}^m(x,z;A,B). \tag{2.6}$$

Proof. Replacing x by $\hat{M}_p$ and y by z in the series definition (1.14) of GHMaP $g_n^m(x,y;A,B)$, we have

$$ g_n^m(\hat{M}_p,z;A,B) = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(\sqrt{2A})^{n-mk}B^kz^k}{(n-mk)!k!}\hat{M}_p^{n-mk}\{1\}. \tag{2.7}$$

Using equation (2.4) in the l.h.s. and equation (1.9) in the r.h.s. of the above equation we get assertion (2.5).

Further, using series expansion (1.11) of 2VgP $p_n(x,y)$ in the r.h.s. of equation (2.5), we get

$$ {}_pg_n^m(x,y,z;A,B) = n!\sum_{k=0}^{[\frac{n-r}{m}]}\sum_{r=0}^{n-mk}\phi_r(y)\frac{(\sqrt{2A})^{n-mk}B^kx^{n-mk-r}z^k}{r!(n-mk-r)!k!}, \tag{2.8}$$

which on using series expansion (1.14) of GHMaP $g_n^m(x,y;A,B)$ and using appropriate indices yields assertion (2.6). □

In order to frame the 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$ within the context of monomiality principle, we first determine multiplicative and derivative operators:

Theorem 2.3. *The 3VgGHMaP* ${}_pg_n^m(x,y,z;A,B)$ *are quasi-monomial under the action of the following multiplicative and derivative operators:*

$$ \hat{M}_{pg} = x\sqrt{2A}+\frac{\Phi'(y,\partial_x)}{\Phi(y,\partial_x)}\sqrt{2A}+mz(\sqrt{2A})^{-(m-1)}B\partial_x^{m-1} \tag{2.9}$$

and

$$ \hat{P}_{pg} = \frac{1}{\sqrt{2A}}\partial_x, \tag{2.10}$$

respectively.

Proof. Consider the identity

$$ \frac{\partial_x}{\sqrt{2A}}\left\{\exp(x\sqrt{2A}\ t)\ \Phi(y,\sqrt{2A}\ t)\ \exp(Bzt^m)\right\} = t\left\{\exp(x\sqrt{2A}\ t)\ \Phi(y,\sqrt{2A}\ t)\ \exp(Bzt^m)\right\}. \tag{2.11}$$

Differentiating equation (2.1) partially with respect to t, we find

$$ \left(x\sqrt{2A}+\frac{\Phi'(y,\sqrt{2A}\ t)}{\Phi(y,\sqrt{2A}\ t)}\sqrt{2A}+mt^{m-1}Bz\right)\{\exp(x\sqrt{2A}\ t)\ \Phi(y,\sqrt{2A}\ t)\ \exp(Bzt^m)\} = \sum_{n=0}^{\infty}{}_pg_{n+1}^m(x,y,z;A,B)\frac{t^n}{n!}. \tag{2.12}$$

Since $\Phi(y,t)$ is an invertible series of t, therefore $\frac{\Phi'(y,t)}{\Phi(y,t)}$ posses power series expansion of t. Thus, in view of the identity (2.11), equation (2.12) becomes

$$\left(x\sqrt{2A}+\frac{\Phi'(y,\partial_x)}{\Phi(y,\partial_x)}\sqrt{2A}+mz(\sqrt{2A})^{-(m-1)}B\partial_x^{m-1}\right)\big\{\exp(x\sqrt{2A}\ t)$$
$$\Phi(y\sqrt{2A}\ t)\exp(Bzt^m)\big\}=\sum_{n=0}^{\infty}{}_pg_{n+1}^m(x,y,z;A,B)\frac{t^n}{n!}. \quad (2.13)$$

Using (2.1) in the l.h.s. of equation (2.13) and rearranging the summation, we get

$$\sum_{n=0}^{\infty}\left(x\sqrt{2A}+\frac{\Phi'(y,\partial_x)}{\Phi(y,\partial_x)}\sqrt{2A}+mz(\sqrt{2A})^{-(m-1)}B\partial_x^{m-1}\right)\{{}_pg_n^m(x,y,z;A,B)\}\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}{}_pg_{n+1}^m(x,y,z;A,B)\frac{t^n}{n!}. \quad (2.14)$$

Now, equating the coefficients of the same powers of t in both sides of the above equation, we find

$$\left(x\sqrt{2A}+\frac{\Phi'(y,\partial_x)}{\Phi(y,\partial_x)}\sqrt{2A}+mz(\sqrt{2A})^{-(m-1)}B\partial_x^{m-1}\right)\{{}_pg_n^m(x,y,z;A,B)\}$$
$$={}_pg_{n+1}^m(x,y,z;A,B), \quad (2.15)$$

which in view of monomiality principle (1.5) (for 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$) yields assertion (2.9).

In order to prove assertion (2.10), we use generating function (2.1) on both sides of the identity (2.11), so that we have

$$\frac{\partial_x}{\sqrt{2A}}\left\{\sum_{n=0}^{\infty}{}_pg_n^m(x,y,z;A,B)\frac{t^n}{n!}\right\}=\sum_{n=1}^{\infty}{}_pg_{n-1}^m(x,y,z;A,B)\frac{t^n}{(n-1)!}. \quad (2.16)$$

Rearranging the summation in the l.h.s. of equation (2.16) and then equating the coefficients of the same powers of t in both sides of the resultant equation, we find

$$\frac{\partial_x}{\sqrt{2A}}\{{}_pg_n^m(x,y,z;A,B)\}=n\ {}_pg_{n-1}^m(x,y,z;A,B), \quad n\geq 1, \quad (2.17)$$

which in view of monomiality principle (1.6) (for 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$) yields assertion (2.10). □

Remark. We remark that equations (2.15) and (2.17) are the differential recurrence relations satisfied by the 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$.

To derive the differential equation for the 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$, we prove the following result:

Theorem 2.4. *The 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$ satisfy the following differential equation:*

$$\left(x\partial_x+\frac{\Phi'(y,\partial_x)}{\Phi(y,\partial_x)}\partial_x+mz(\sqrt{2A})^{-m}B\partial_x^m-n\right){}_pg_n^m(x,y,z;A,B)=0. \quad (2.18)$$

Proof. Using expressions (2.9) and (2.10) and in view of monomiality principle (1.7) (for 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$), we get assertion (2.18). □

Theorem 2.5. *The following operational representation for 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$ holds true:*

$$ {}_pg_n^m(x,y,z;A,B)=\exp(z(\sqrt{2A})^{-m}B\partial_x^m)\{(\sqrt{2A})^n p_n(x,y)\}. \tag{2.19}$$

Proof. Using operational representation (1.23) of GHMaP $g_n^m(x,y;A,B)$ in the r.h.s of the series definition (2.6) of 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$, we get

$$ {}_pg_n^m(x,y,z;A,B)=n!\sum_{r=0}^{n}\frac{\phi_r(y)}{r!(n-r)!}\exp(z(\sqrt{2A})^{-m}B\ \partial_x^m)\ x^{n-r}(\sqrt{2A})^n, \tag{2.20}$$

which on using the series representation (1.11) of 2VgP $p_n(x,y)$ on the r.h.s, gives assertion (2.19). □

In view of Table 2, taking suitable values of the matrices and indices in equations (2.1), (2.5), (2.6), (2.9), (2.10), (2.18) and (2.19), we can find the generating function and other properties for the mixed special polynomials related to 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$. We present the generating functions for these polynomials in Table 3.

TABLE 3. Certain members belonging to the 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$.

S.No.	Values of the matrices and indices	Relation between the 3VgGHMaP ${}_p\mathbf{g}_\mathbf{n}^\mathbf{m}(\mathbf{x},\mathbf{y},\mathbf{z};\mathbf{A},\mathbf{B})$ & its special cases	Name of the resultant mixed special polynomials	Generating function of the resultant special polynomials
I.	$A\to\frac{r}{2}A$, $B=I\in\mathbb{C}^{N\times N}$	${}_pg_n^m(x,y,z;\frac{r}{2}A,I)$ $={}_pH_{n,r}^{(m)}(x,y,z;A)$	3-variable general-3-index Hermite matrix polynomials (3Vg3IHMaP)	$\exp\left(x\sqrt{rA}\ t+zt^m\right)\Phi\left(y,\sqrt{rA}\ t\right)$ $=\sum_{n=0}^{\infty}{}_pH_{n,r}^{(m)}(x,y,z,A)\frac{t^n}{n!}$
II.	$A\to\frac{m}{2}A$, $B=-I\in\mathbb{C}^{N\times N}$	${}_pg_n^m(x,y,z;\frac{m}{2}A,-I)$ $={}_pH_{n,m}(x,y,z,A)$	3-variable general-generalized Hermite matrix polynomials (3VgGHMaP)	$\exp\left(x\sqrt{mA}\ t-zt^m\right)\Phi\left(y,\sqrt{mA}\ t\right)$ $=\sum_{n=0}^{\infty}{}_pH_{n,m}(x,y,z,A)\frac{t^n}{n!}$
III.	$A=\frac{1}{2}\in\mathbb{C}^{1\times 1}$, $B=1\in\mathbb{C}^{1\times 1}$	${}_pg_n^m(x,y,z;\frac{1}{2},1)$ $={}_pH_n^{(m)}(x,y,z)$	3-variable general -Gould-Hopper polynomials (3VgGHP)	$\exp\left(xt+zt^m\right)\Phi\left(y,t\right)$ $=\sum_{n=0}^{\infty}{}_pH_n^{(m)}(x,y,z)\frac{t^n}{n!}$
IV.	$B=-I\in\mathbb{C}^{N\times N}$, $m=2$	${}_pg_n^2(x,y,z;A,-I)$ $={}_pH_n(x,y,z,A)$	3-variable general-Hermite matrix polynomials (3VgHMaP)	$\exp\left(x\sqrt{2A}\ t-zt^2\right)\Phi\left(y,\sqrt{2A}\ t\right)$ $=\sum_{n=0}^{\infty}{}_pH_n(x,y,z,A)\frac{t^n}{n!}$
V.	$A\to\frac{1}{2}\in\mathbb{C}^{1\times 1}$, $B=1\in\mathbb{C}^{1\times 1}$, $m=2$	${}_pg_n^2(x,y,z;\frac{1}{2},1)$ $={}_pH_n(x,y,z)$	3-variable general-Hermite polynomials (3VgHP)	$\exp\left(xt+zt^2\right)\Phi\left(y,t\right)$ $=\sum_{n=0}^{\infty}{}_pH_n(x,y,z)\frac{t^n}{n!}$

In the next section, examples of some members belonging to the 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$ are considered.

3. EXAMPLES

The family of 2VgP $p_n(x,y)$ contains a number of important special polynomials of two variable. Certain members belonging to the family of 2VgP $p_n(x,y)$ are considered in Table 1. We note that corresponding to each member belonging to the 2VgP $p_n(x,y)$, there exist a new special polynomial belonging to the family of 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$. Thus, by making suitable choice for the function $\Phi(y,t)$ in the results derived in the previous section, we get the properties of these new special polynomials.

We consider the following examples:

Example 3.1. Taking $\Phi(y,t) = e^{yt^s}$ (for which the 2VgP $p_n(x,y)$ reduce to the GHP $H_n^{(s)}(x,y)$ of order s) in the l.h.s. of generating function (2.1), we find that the resultant Gould-Hopper-Gould-Hopper matrix polynomials (GHGHMaP), denoted by ${}_{H^{(s)}}g_n^m(x,y,z;A,B)$ are defined by the following generating function:

$$\exp(x\sqrt{2A}\ t + y(\sqrt{2A}\ t)^s)\exp(Bzt^m) = \sum_{n=0}^{\infty} {}_{H^{(s)}}g_n^m(x,y,z;A,B)\frac{t^n}{n!}. \tag{3.1}$$

The series definition and other results for the GHGHMaP ${}_{H^{(s)}}g_n^m(x,y,z;A,B)$ are given in Table 4.

TABLE 4. Results for the GHGHMaP ${}_{H^{(s)}}g_n^m(x,y,z;A,B)$.

S.No.	Results	Expressions
1.	**Series definition**	${}_{H^{(s)}}g_n^m(x,y,z;A,B) = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(\sqrt{2A})^{n-mk}B^kz^k}{(n-mk)!k!}H_{n-mk}^{(s)}(x.y)$
2.	**Multiplicative and derivative operators**	$\hat{M}_{H^{(s)}g} := x\sqrt{2A} + mz(\sqrt{2A})^{-(m-1)}B\partial_x^{m-1} + \sqrt{2A}sy\partial_x^{s-1}$ $\hat{P}_{H^{(s)}g} := \frac{1}{\sqrt{2A}}\partial_x$
3.	**Differential equation**	$(x\partial_x + sy\partial_x^s + mz(\sqrt{2A})^{-m}B\partial_x^m - n)\,{}_{H^{(s)}}g_n^m(x,y,z;A,B) = 0$
4.	**Operational rule**	${}_{H^{(s)}}g_n^m(x,y,z;A,B) = \exp\left(z(\sqrt{2A})^{-m}B\partial_x^m\right)\{(\sqrt{2A})^nH_n^{(s)}(x,y)\}$

Remark. Since for $s = 2$, the GHP $H_n^{(s)}(x,y)$ reduce to the 2VHKdFP $H_n(x,y)$. Therefore, taking $s = 2$ in equation (3.1), we get the following generating function for the Hermite-Gould-Hopper matrix polynomials (HGHMaP) denoted by ${}_Hg_n^m(x,y,z;A,B)$:

$$\exp(x\sqrt{2A}\ t + y(\sqrt{2A}\ t)^2)\exp(Bzt^m) = \sum_{n=0}^{\infty} {}_Hg_n^m(x,y,z;A,B)\frac{t^n}{n!}. \tag{3.2}$$

The series definition and other results for the HGHMaP ${}_Hg_n^m(x,y,z;A,B)$ can be obtained by taking $s = 2$ in the results given in Table 4.

In view of Table 2, taking suitable values of the matrices and indices in the results of GHGHMaP ${}_{H^{(s)}}g_n^m(x,y,z;A,B)$, we can find the corresponding results for the mixed special polynomials related to ${}_{H^{(s)}}g_n^m(x,y,z;A,B)$. We use the suitable notations for these polynomials and present their generating functions in Table 5.

TABLE 5. Certain members belonging to the GHGHMaP ${}_{H^{(s)}}g_n^m(x,y,z;A,B)$.

S.No.	Relation between the GHGHMaP ${}_{\mathbf{H^{(s)}}}\mathbf{g_n^m}(\mathbf{x},\mathbf{y},\mathbf{z};\mathbf{A},\mathbf{B})$ & its special cases	Name of the resultant mixed special polynomials	Generating function of the resultant special polynomials
I.	${}_{H^{(s)}}g_n^m(x,y,z;\frac{r}{2}A,I)$ $={}_{H^{(s)}}H_{n,r}^{(m)}(x,y,z;A)$	Gould-Hopper-3-index Hermite matrix polynomials (GH3IHMaP)	$\exp(x\sqrt{rA}\ t+y(\sqrt{rA}\ t)^s+zt^m)$ $=\sum_{n=0}^{\infty}{}_{H^{(s)}}H_{n,r}^{(m)}(x,y,z,A)\frac{t^n}{n!}$
II.	${}_{H^{(s)}}g_n^m(x,y,z;\frac{m}{2}A,-I)$ $={}_{H^{(s)}}H_{n,m}(x,y,z,A)$	Gould-Hopper-generalized Hermite matrix polynomials (GHGHMaP)	$\exp(x\sqrt{mA}\ t+y(\sqrt{mA}\ t)^s-zt^m)$ $=\sum_{n=0}^{\infty}{}_{H^{(s)}}H_{n,m}(x,y,z,A)\frac{t^n}{n!}$
III.	${}_{H^{(s)}}g_n^m(x,y,z;\frac{1}{2},1)$ $={}_{H^{(s)}}H_n^{(m)}(x,y,z)$	2-iterated Gould-Hopper polynomials (2IGHP)	$\exp(xt+yt^s+zt^m)$ $=\sum_{n=0}^{\infty}{}_{H^{(s)}}H_n^{(m)}(x,y,z)\frac{t^n}{n!}$
IV.	${}_{H^{(s)}}g_n^2(x,y,z;A,-I)$ $={}_{H^{(s)}}H_n(x,y,z,A)$	Gould-Hopper-Hermite matrix polynomials(GHHMaP)	$\exp(x\sqrt{2A}\ t+y(\sqrt{2A}\ t)^s-zt^2)$ $=\sum_{n=0}^{\infty}{}_{H^{(s)}}H_n(x,y,z,A)\frac{t^n}{n!}$
V.	${}_{H^{(s)}}g_n^2(x,y,z;\frac{1}{2},1)$ $={}_{H^{(s)}}H_n(x,y,z)$	Gould-Hopper-Hermite polynomials (GHHP)	$\exp(xt+yt^s+zt^2)$ $=\sum_{n=0}^{\infty}{}_{H^{(s)}}H_n(x,y,z)\frac{t^n}{n!}$

It is also important to observe that taking $s=2$ in the generating functions (Table 5) of the mixed special polynomials with $H_n^{(s)}(x,y)$ as base, we obtain the results for the corresponding mixed special polynomials with $H_n(x,y)$ as base.

Example 3.2. Taking $\Phi(y,t)=C_0(-yt^s)$ (for which the 2VgP $p_n(x,y)$ reduce to the 2VGLP ${}_sL_n(y,x)$) in the l.h.s. of generating function (2.1), we find that the resultant generalized Laguerre-Gould-Hopper matrix polynomials (GLGHMaP), denoted by ${}_{{}_sL}g_n^m(x,y,z;A,B)$ are defined by the following generating function:

$$\exp(x\sqrt{2A}\ t)C_0(-y(\sqrt{2A}\ t)^s)\exp(Bzt^m)=\sum_{n=0}^{\infty}{}_{{}_sL}g_n^m(x,y,z;A,B)\frac{t^n}{n!}. \tag{3.3}$$

The series definition and other results for the GLGHMaP ${}_{{}_sL}g_n^m(x,y,z;A,B)$ are given in Table 6.

TABLE 6. Results for the GLGHMaP ${}_{sL}g_n^m(x,y,z;A,B)$.

S.No.	Results	Expressions
1.	**Series definition**	${}_{{}_sL}g_n^m(x,y,z;A,B)=n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(\sqrt{2A})^{n-mk}B^kz^k}{(n-mk)!k!}{}_sL_{n-mk}(y,x)$
2.	**Multiplicative and derivative operators**	$\hat{M}_{{}_sLg}:=x\sqrt{2A}+mz(\sqrt{2A})^{-(m-1)}B\partial_x^{m-1}+\sqrt{2A}\ s\partial_y^{-1}\partial_x^{s-1}$ $\hat{P}_{{}_sLg}::=\frac{1}{\sqrt{2A}}\partial_x$
3.	**Differential equation**	$(x\partial_x\partial_y+s\partial_x^s+mz(\sqrt{2A})^{-m}B\partial_x^m\partial_y-n\partial_y)_{{}_sL}g_n^m(x,y,z;A,B)=0$
4.	**Operational rule**	${}_{{}_sL}g_n^m(x,y,z;A,B)=\exp(z(\sqrt{2A})^{-m}B\partial_x^m)\{(\sqrt{2A})^n{}_sL_n(y,x)\}$

Remark. Since for $s = 1$ and $y \to -y$, the 2VGLP ${}_sL_n(y,x)$ reduce to the 2VLP $L_n(y,x)$. Therefore, taking $s = 1$ and $y \to -y$ in equation (3.3), we get the following generating function for the Laguerre-Gould-Hopper matrix polynomials (LGHMaP) denoted by ${}_Lg_n^m(x,y,z;A,B)$:

$$\exp(x\sqrt{2A}\ t)C_0(y\sqrt{2A}\ t)\exp(Bzt^m) = \sum_{n=0}^{\infty} {}_Lg_n^m(x,y,z;A,B)\frac{t^n}{n!}. \tag{3.4}$$

The series definition and other results for the LGHMaP ${}_Lg_n^m(x,y,z;A,B)$ can be obtained by taking $s = 1$ and $y \to -y$ in the results given in Table 6.

In view of Table 2, taking suitable values of the matrices and indices in the results of the GLGHMaP ${}_{{}_sL}g_n^m(x,y,z;A,B)$, we can find the corresponding results for the mixed special polynomials related to ${}_{{}_sL}g_n^m(x,y,z;A,B)$. We use the suitable notations for these polynomials and present their generating functions in Table 7.

Table 7. Certain members belonging to the GLGHMaP ${}_{{}_sL}g_n^m(x,y,z;A,B)$.

S.No.	Relation between the GLGHMaP ${}_{{}_sL}\mathbf{g}_n^m(\mathbf{x},\mathbf{y},\mathbf{z};\mathbf{A},\mathbf{B})$	Name of the resultant mixed special polynomials & its special cases	Generating function of the resultant special polynomials
I.	${}_{{}_sL}g_n^m(x,y,z;\frac{r}{2}A,I) = {}_{{}_sL}H_{n,r}^{(m)}(x,y,z;A)$	Generalized Laguerre-3-index Hermite matrix polynomials (GL3IHMaP)	$\exp(x\sqrt{rA}\ t+zt^m)C_0(-y(\sqrt{rA}\ t)^s) = \sum_{n=0}^{\infty} {}_{{}_sL}H_{n,r}^{(m)}(x,y,z,A)\frac{t^n}{n!}$
II.	${}_{{}_sL}g_n^m(x,y,z;\frac{m}{2}A,-I) = {}_{{}_sL}H_{n,m}(x,y,z,A)$	Generalized Laguerre-generalized Hermite matrix polynomials (GLÇHMaP)	$\exp(x\sqrt{mA}\ t-zt^m)C_0(-y(\sqrt{mA}\ t)^s) = \sum_{n=0}^{\infty} {}_{{}_sL}H_{n,m}(x,y,z,A)\frac{t^n}{n!}$
III.	${}_{{}_sL}g_n^m(x,y,z;\frac{1}{2},1) = {}_{{}_sL}H_n^{(m)}(x,y,z)$	Generalized Laguerre-Gould-Hopper polynomials (GLGHP)	$\exp(xt+zt^m)C_0(-yt^s) = \sum_{n=0}^{\infty} {}_{{}_sL}H_n^{(m)}(x,y,z)\frac{t^n}{n!}$
IV.	${}_{{}_sL}g_n^2(x,y,z;A,-I) = {}_{{}_sL}H_n(x,y,z,A)$	Generalized Laguerre-Hermite matrix polynomials (GLHMaP)	$\exp(x\sqrt{2A}\ t-zt^2)C_0(-y(\sqrt{2A}\ t)^s) = \sum_{n=0}^{\infty} {}_{{}_sL}H_n(x,y,z,A)\frac{t^n}{n!}$
V.	${}_{{}_sL}g_n^2(x,y,z;\frac{1}{2},1) = {}_{{}_sL}H_n(x,y,z)$	Generalized Laguerre-Hermite polynomials (GLHP)	$\exp(xt+zt^2)C_0(-yt^s) = \sum_{n=0}^{\infty} {}_{{}_sL}H_n(x,y,z)\frac{t^n}{n!}$

It is also important to observe that taking $s = 1$ and $y \to -y$ in the generating functions (Table 7) of the mixed special polynomials with ${}_sL_n(y,x)$ as base, we obtain the results for the corresponding mixed special polynomials with $L_n(y,x)$ as base.

Example 3.3. Taking $\Phi(y,t) = \frac{1}{1-yt^r}$ (for which the 2VgP $p_n(x,y)$ reduce to the 2VTEP (of order r) $e_n^{(r)}(x,y)$) in the l.h.s. of generating function (2.1), we find that the resultant 3-variable truncated exponential-Gould-Hopper matrix polynomials (3VTEGHMaP), denoted by ${}_{e^{(r)}}g_n^m(x,y,z;A,B)$ are defined by the following generating function:

$$\frac{\exp(x\sqrt{2A}\ t)\exp(Bzt^m)}{1-y(\sqrt{2A}\ t)^r} = \sum_{n=0}^{\infty} {}_{e^{(r)}}g_n^m(x,y,z;A,B)\frac{t^n}{n!}. \tag{3.5}$$

The series definition and other results for the 3VTEGHMaP ${}_{e^{(r)}}g_n^m(x,y,z;A,B)$ are given in Table 8.

Table 8. Results for the 3VTEGHMaP ${}_{e^{(r)}}g_n^m(x,y,z;A,B)$.

S.No.	Results	Expressions
1.	**Series definition**	${}_{e^{(r)}}g_n^m(x,y,z;A,B) = n!\sum_{k=0}^{[\frac{n}{m}]} \frac{(\sqrt{2A})^{n-mk}B^k z^k}{(n-mk)!k!} e_{n-mk}^{(r)}(x,y)$
2.	**Multiplicative and derivative operators**	$\hat{M}_{{}_{e^{(r)}}g} := x\sqrt{2A} + mz(\sqrt{2A})^{-(m-1)}B\partial_x^{m-1} + \sqrt{2A}\ ry\partial_y y\partial_x^{r-1}$ $\hat{P}_{{}_{e^{(r)}}g} := \frac{1}{\sqrt{2A}}\partial_x$
3.	**Differential equation**	$(x\partial_x + ry\partial_y y\partial_x^r + mz(\sqrt{2A})^{-m}B\partial_x^m - n)\,{}_{e^{(r)}}g_n^m(x,y,z;A,B) = 0$
4.	**Operational rule**	${}_{e^{(r)}}g_n^m(x,y,z;A,B) = \exp\left(z(\sqrt{2A})^{-m}B\partial_x^m\right)\{(\sqrt{2A})^n e_n^{(r)}(x,y)\}$

In view of Table 2, taking suitable values of the matrices and indices in the results of the 3VTEGHMaP ${}_{e^{(r)}}g_n^m(x,y,z;A,B)$, we can find the corresponding results for the mixed special polynomials related to ${}_{e^{(r)}}g_n^m(x,y,z;A,B)$. We use the suitable notations for these polynomials and present their generating functions in Table 9.

Table 9. Certain members belonging to the 3VTEGHMaP ${}_{e^{(r)}}g_n^m(x,y,z;A,B)$.

S.No.	Relation between the 3VTEGHMaP $\mathbf{{}_{e^{(r)}}g_n^m(x,y,z;A,B)}$ & its special cases	Name of the resultant mixed special polynomials	Generating function of the resultant special polynomials
I.	${}_{e^{(i)}}g_n^m(x,y,z;\frac{r}{2}A,I)$ $= {}_{e^{(i)}}H_{n,r}^{(m)}(x,y,z;A)$	3-variable truncated exponential-3-index Hermite matrix polynomials (3VTE3IHMaP)	$\exp\left(x\sqrt{rA}\ t + zt^m\right)(1 - y(\sqrt{rA}\ t)^i)^{-1}$ $= \sum_{n=0}^{\infty} {}_{e^{(i)}}H_{n,r}^{(m)}(x,y,z,A)\frac{t^n}{n!}$
II.	${}_{e^{(r)}}g_n^m(x,y,z;\frac{m}{2}A,-I)$ $= {}_{e^{(r)}}H_{n,m}(x,y,z,A)$	3-variable truncated exponential-generalized Hermite matrix polynomials (3VTEGHMaP)	$\exp\left(x\sqrt{mA}\ t - zt^m\right)(1 - y(\sqrt{mA}\ t)^r)^{-1}$ $= \sum_{n=0}^{\infty} {}_{e^{(r)}}H_{n,m}(x,y,z,A)\frac{t^n}{n!}$
III.	${}_{e^{(r)}}g_n^m(x,y,z;\frac{1}{2},1)$ $= {}_{e^{(r)}}H_n^{(m)}(x,y,z)$	3-variable truncated exponential-Gould-Hopper polynomials (3VTEGHP)	$\exp(xt + zt^m)(1 - yt^r)^{-1}$ $=\sum_{n=0}^{\infty} {}_{e^{(r)}}H_n^{(m)}(x,y,z)\frac{t^n}{n!}$
IV.	${}_{e^{(r)}}g_n^2(x,y,z;A,-I)$ $= {}_{e^{(r)}}H_n(x,y,z,A)$	3-variable truncated exponential-Hermite matrix polynomials (3VTEHMaP)	$\exp\left(x\sqrt{2A}\ t - zt^2\right)(1 - y(\sqrt{2A}\ t)^r)^{-1}$ $= \sum_{n=0}^{\infty} {}_{e^{(r)}}H_n(x,y,z,A)\frac{t^n}{n!}$
V.	${}_{e^{(r)}}g_n^2(x,y,z;\frac{1}{2},1)$ $= {}_{e^{(r)}}H_n(x,y,z)$	3-variable truncated exponential-Hermite polynomials (3VTEHP)	$\exp\left(xt + zt^2\right)(1 - yt^r)^{-1}$ $= \sum_{n=0}^{\infty} {}_{e^{(r)}}H_n(x,y,z)\frac{t^n}{n!}$

Example 3.4. Taking $\Phi(y,t) = \frac{t}{\exp(t)-1}\exp(yt^j)$ (for which the 2VgP $p_n(x,y)$ reduce to the 2DBP $B_n^{(j)}(x,y)$) in the l.h.s. of generating function (2.1), we find that the resultant 2D Bernoulli-Gould-Hopper matrix polynomials (2DBGHMaP),

denoted by ${}_{B^{(j)}}g_n^m(x,y,z;A,B)$ are defined by the following generating function:

$$\frac{\sqrt{2A}\,t}{\exp(\sqrt{2A}\,t)-1}\exp(x\sqrt{2A}\,t+y(\sqrt{2A}t)^j)\exp(Bzt^m)=\sum_{n=0}^{\infty}{}_{B^{(j)}}g_n^m(x,y,z;A,B)\frac{t^n}{n!}. \tag{3.6}$$

The series definition and other results for the (2DBGHMaP) ${}_{B^{(j)}}g_n^m(x,y,z;A,B)$ are given in Table 10.

TABLE 10. Results for the 2DBGHMaP ${}_{B^{(j)}}g_n^m(x,y,z;A,B)$.

S.No.	**Results**	**Expressions**
1.	**Series definition**	${}_{B^{(j)}}g_n^m(x,y,z;A,B)=n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(\sqrt{2A})^{n-mk}B^kz^k}{(n-mk)!k!}B_{n-mk}^{(j)}(x,y)$
2.	**Multiplicative and derivative operators**	$\hat{M}_{B^{(j)}g}:=\sqrt{2A}\,(x+\partial_x^{-1}+jy\partial_x^{j-1}+(1-\exp(\partial_x))^{-1})$ $+mz(\sqrt{2A})^{-(m-1)}B\partial_x^{m-1}$ $\hat{P}_{B^{(j)}g}:=\frac{1}{\sqrt{2A}}\partial_x$
3.	**Differential equation**	$(1+x\partial_x+yj\partial_x^j+(1-\exp(\partial_x))^{-1}\partial_x$ $+mz(\sqrt{2A})^{-m}B\partial_x^m-n)\,{}_{B^{(j)}}g_n^m(x,y,z;A,B)=0$
4.	**Operational rule**	${}_{B^{(j)}}g_n^m(x,y,z;A,B)=\exp\left(z(\sqrt{2A})^{-m}B\partial_x^m\right)\{(\sqrt{2A})^nB_n^{(j)}(x,y)\}$

In view of Table 2, taking suitable values of the matrices and indices in the results of the 2DBGHMaP ${}_{B^{(j)}}g_n^m(x,y,z;A,B)$, we can find the corresponding results for the mixed special polynomials related to ${}_{B^{(j)}}g_n^m(x,y,z;A,B)$. We use the suitable notations for these polynomials and present their generating functions in Table 11.

TABLE 11. Certain members belonging to the 2DBGHMaP ${}_{B^{(j)}}g_n^m(x,y,z;A,B)$.

S.No.	**Relation between the 2DBGHMaP $\mathbf{{}_{B^{(j)}}g_n^m(x,y,z;A,B)}$ & its special cases**	**Name of the resultant mixed special polynomials**	**Generating function of the resultant special polynomials**
I.	${}_{B^{(j)}}g_n^m(x,y,z;\frac{r}{2}A,I)$ $={}_{B^{(j)}}H_{n,r}^{(m)}(x,y,z;A)$	2D Bernoulli-3-index Hermite matrix polynomials (2DB3IHMaP)	$\frac{\sqrt{rA}\;t\exp\left(x\sqrt{rA}\;t+y(\sqrt{rA}\;t)^j+zt^m\right)}{\exp(\sqrt{rA}\;t)-1}$ $=\sum_{n=0}^{\infty}{}_{B^{(j)}}H_{n,r}^{(m)}(x,y,z,A)\frac{t^n}{n!}$
II.	${}_{B^{(j)}}g_n^m(x,y,z;\frac{m}{2}A,-I)$ $={}_{B^{(j)}}H_{n,m}(x,y,z,A)$	2D Bernoulli-generalized Hermite matrix polynomials (2DBGHMaP)	$\frac{\sqrt{mA}\;t\exp\left(x\sqrt{mA}\;t+y(\sqrt{mA}\;t)^j-zt^m\right)}{\exp(\sqrt{mA}\;t)-1}$ $=\sum_{n=0}^{\infty}{}_{B^{(j)}}H_{n,m}(x,y,z,A)\frac{t^n}{n!}$
III.	${}_{B^{(j)}}g_n^m(x,y,z;\frac{1}{2},1)$ $={}_{B^{(j)}}H_n^{(m)}(x,y,z)$	2D Bernoulli-Gould-Hopper polynomials (2DBGHP)	$\frac{t\exp\left(xt+yt^j+zt^m\right)}{\exp(t)-1}$ $=\sum_{n=0}^{\infty}{}_{B^{(j)}}H_n^{(m)}(x,y,z)\frac{t^n}{n!}$

S.No.	Relation between the 2DBGHMaP ${}_{B^{(j)}}g^{m}_{n}(x, y, z; A, B)$ & its special cases	Name of the resultant mixed special polynomials	Generating function of the resultant special polynomials
IV.	${}_{B^{(j)}}g^{2}_{n}(x,y,z;A,-I)$ $= {}_{B^{(j)}}H_n(x,y,z,A)$	2D Bernoulli-Hermite matrix polynomials(2DBHMaP)	$\frac{\sqrt{2A}\ t\exp\left(x\sqrt{2A}\ t+y(\sqrt{2A}\ t)^j-zt^2\right)}{\exp(\sqrt{2A}\ t)-1}$ $=\sum_{n=0}^{\infty} {}_{B^{(j)}}H_n(x,y,z,A)\frac{t^n}{n!}$
V.	${}_{B^{(j)}}g^{2}_{n}(x,y,z;\frac{1}{2},1)$ $= {}_{B^{(j)}}H_n(x,y,z)$	2D Bernoulli-Hermite polynomials (2DBHP)	$\frac{t\exp\left(xt+yt^j+zt^2\right)}{\exp(t)-1}$ $=\sum_{n=0}^{\infty} {}_{B^{(j)}}H_n(x,y,z)\frac{t^n}{n!}$

Example 3.5. Taking $\Phi(y,t) = \frac{2}{\exp(t)+1}\exp(yt^j)$ (for which the 2VgP $p_n(x,y)$ reduce to the 2DEP $E_n^{(j)}(x,y)$) in the l.h.s. of generating function (2.1), we find that the 2D Euler-Gould-Hopper matrix polynomials (2DEGHMaP), denoted by ${}_{E^{(j)}}g^m_n(x,y,z;A,B)$ are defined by the following generating function:

$$\frac{2}{\exp(\sqrt{2A}\ t)+1}\exp(x\sqrt{2A}\ t+y(\sqrt{2A})^j)\exp(Bzt^m)=\sum_{n=0}^{\infty} {}_{E^{(j)}}g^m_n(x,y,z;A,B)\frac{t^n}{n!}. \tag{3.7}$$

The series definition and other results for the 2DEGHMaP ${}_{E^{(j)}}g^m_n(x,y,z;A,B)$ are given in Table 12.

TABLE 12. Results for the 2DEGHMaP ${}_{E^{(j)}}g^m_n(x,y,z;A,B)$.

S.No.	Results	Expressions
1.	Series definition	${}_{E^{(j)}}g^m_n(x,y,z;A,B)=n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(\sqrt{2A})^{n-mk}B^kz^k}{(n-mk)!k!}E^{(j)}_{n-mk}(x,y)$
2.	Multiplicative and derivative operators	$\hat{M}_{{}_{E^{(j)}}g} := \sqrt{2A}\ (x+jy\partial_x^{j-1}-(1+\exp(\partial_x))^{-1}$ $+mz(\sqrt{2A})^{-m}B\partial_x^{m-1})$ $\hat{P}_{{}_{E^{(j)}}g} := \frac{1}{\sqrt{2A}}\partial_x$
3.	Differential equation	$(x\partial_x+jy\partial_x^j-(1+\exp(\partial_x))^{-1}\partial_x$ $+mz(\sqrt{2A})^{-m}B\partial_x^m-n)\,{}_{E^{(j)}}g^m_n(x,y,z;A,B)=0$
4.	Operational rule	${}_{E^{(j)}}g^m_n(x,y,z;A,B)=\exp\left(z(\sqrt{2A})^{-m}B\partial_x^m\right)\{(\sqrt{2A})^nE^{(j)}_n(x,y)\}$

In view of Table 2, taking suitable values of the matrices and indices in the results of the 2DEGHMaP ${}_{E^{(j)}}g^m_n(x,y,z;A,B)$, we can find the corresponding results for the mixed special polynomials related to ${}_{E^{(j)}}g^m_n(x,y,z;A,B)$. We use the suitable notations for these polynomials and present their generating functions in Table 13.

TABLE 13. Certain members belonging to the 2DEGHMaP ${}_{E^{(j)}}g_n^m(x,y,z;A,B)$.

S.No.	Relation between the 2DEGHMaP ${}_{\mathbf{E}^{(\mathbf{j})}}\mathbf{g}_\mathbf{n}^\mathbf{m}(\mathbf{x},\mathbf{y},\mathbf{z};\mathbf{A},\mathbf{B})$ & its special cases	Name of the resultant mixed special polynomials	Generating function of the resultant special polynomials
I.	${}_{E^{(j)}}g_n^m(x,y,z;\frac{r}{2}A,I) = {}_{E^{(j)}}H_{n,r}^{(m)}(x,y,z;A)$	2D Euler-3-index Hermite matrix polynomials (2DE3IHMaP)	$\frac{2}{\exp(\sqrt{rA}\ t)+1}\exp\left(x\sqrt{rA}\ t+y(\sqrt{rA}\ t)^j+zt^m\right) = \sum_{n=0}^{\infty}{}_{E^{(j)}}H_{n,r}^{(m)}(x,y,z,A)\frac{t^n}{n!}$
II.	${}_{E^{(j)}}g_n^m(x,y,z;\frac{m}{2}A,-I) = {}_{E^{(j)}}H_{n,m}(x,y,z,A)$	2D Euler-generalized Hermite matrix polynomials (2DEGHMaP)	$\frac{2}{\exp(\sqrt{mA}\ t)+1}\exp\left(x\sqrt{mA}\ t+y(\sqrt{mA}\ t)^j-zt^m\right) = \sum_{n=0}^{\infty}{}_{E^{(j)}}H_{n,m}(x,y,z,A)\frac{t^n}{n!}$
III.	${}_{E^{(j)}}g_n^m(x,y,z;\frac{1}{2},1) = {}_{E^{(j)}}H_n^{(m)}(x,y,z)$	2D Euler-Gould-Hopper polynomials (2DEGHP)	$\frac{2}{\exp(t)+1}\exp\left(xt+yt^j+zt^m\right) = \sum_{n=0}^{\infty}{}_{E^{(j)}}H_n^{(m)}(x,y,z)\frac{t^n}{n!}$
IV.	${}_{E^{(j)}}g_n^2(x,y,z;A,-I) = {}_{E^{(j)}}H_n(x,y,z,A)$	2D Euler-Hermite matrix polynomials (2DEHMaP)	$\frac{2}{\exp(\sqrt{2A}\ t)+1}\exp\left(x\sqrt{2A}\ t+y(\sqrt{2A}\ t)^j-zt^2\right) = \sum_{n=0}^{\infty}{}_{E^{(j)}}H_n(x,y,z,A)\frac{t^n}{n!}$
V.	${}_{E^{(j)}}g_n^2(x,y,z;\frac{1}{2},1) = {}_{E^{(j)}}H_n(x,y,z)$	2D Euler-Hermite polynomials (2DEHP)	$\frac{2}{\exp(\sqrt{A}\ t)+1}\exp\left(xt+yt^j+zt^2\right) = \sum_{n=0}^{\infty}{}_{E^{(j)}}H_n(x,y,z)\frac{t^n}{n!}$

In the next section, bilateral and bilinear generating matrix functions for the newly introduced matrix family are explored.

4. BILATERAL AND BILINEAR GENERATING MATRIX FUNCTIONS

In this section, we derive several families of bilateral and bilinear generating functions for 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$. We begin by stating the following theorem.

Theorem 4.1. *Corresponding to an identically non-vanishing function $\Omega_\nu(q_1,\cdots,q_r)$ consisting of r complex variables $q_1,\cdots,q_r$ $(r\in\mathbb{N})$ and of complex order ν, let*

$$\Lambda_{\nu,\eta}(q_1,\cdots,q_r;\psi) := \sum_{k=0}^{\infty}a_k\Omega_{\nu+\eta k}(q_1,\cdots,q_r)\psi^k, \quad (a_k\neq 0,\ \nu,\eta\in\mathbb{C}) \tag{4.1}$$

and

$$\Theta_{n,s,\nu,\eta}^m(x,y,z;q_1,\cdots,q_r;\xi) := \sum_{k=0}^{[\frac{n}{s}]}\frac{a_k}{(n-sk)!}{}_pg_{n-sk}^m(x,y,z;A,B)\Omega_{\nu+\eta k}(q_1,\cdots,q_r)\xi^k, \quad (n,s\in\mathbb{N}). \tag{4.2}$$

Then we have

$$\sum_{n=0}^{\infty}\Theta_{n,s,\nu,\eta}^m\left(x,y,z,q_1,\cdots q_r;\frac{\gamma}{t^s}\right)t^n = \exp(x\sqrt{2A}\ t)\Phi(y,\sqrt{2A}\ t)\exp(Bzt^m)\Lambda_{\nu,\eta}(q_1,\cdots q_r;\gamma), \tag{4.3}$$

provided that each member of (4.3) exists.

Proof. Let us denote the l.h.s. of equation (4.3) by T. Then from (4.2), we get

$$T=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{s}]}a_k\ {}_pg_{n-sk}^m(x,y,z;A,B)\Omega_{\nu+\eta k}(q_1,\cdots q_r)\gamma^k\frac{t^{n-sk}}{(n-sk)!}. \tag{4.4}$$

Now using the relation [15]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}A(k,n-mk) \tag{4.5}$$

gives

$$\begin{aligned} T &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\ {}_pg_n^m(x,y,z;A,B)\Omega_{\nu+\eta k}(q_1,\cdots,q_r)\gamma^k\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}{}_pg_n^m(x,y,z;A,B)\frac{t^n}{n!}\sum_{k=0}^{\infty}a_k\Omega_{\nu+\eta k}(q_1,\cdots,q_r)\gamma^k, \end{aligned}$$

which on using equation (2.1) and (4.1) gives assertion (4.3). □

In order to give some applications of Theorem 4.1, we consider the multivariable function $\Omega_{\nu+\eta k}(q_1,\cdots q_r)$ $(k\in\mathbb{N}_0, r\in\mathbb{N})$ in terms of the functions of one or more variables. For example, setting $r=2$ and $\Omega_{\nu+\eta k}(u,v)=C_{\nu+\eta k}^{(\mu)}(u,v;\alpha,D)$ in Theorem 4.1, where $C_{\nu+\eta k}^{(\mu)}(u,v;\alpha,D)$ denotes the 2-variable 1-parameter Gegenbauer matrix polynomials (2V1PGeMaP) generated by [17]

$$(\alpha I-xt\sqrt{2A}+yt^2I)^{-\mu}=\sum_{n=0}^{\infty}C_n^{(\mu)}(x,y;\alpha,A)t^n,\quad(\mu\neq0), \tag{4.6}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ satisfying condition (1.13). Then we obtain the following result which provides a class of bilateral generating matrix relations for 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$ and 2V1PGeMaP $C_n^{(\mu)}(x,y;\alpha,A)$.

Corollary 4.2. *Let*

$$\Lambda_{\nu,\eta}(u,v;\psi):=\sum_{k=0}^{\infty}a_kC_{\nu+\eta k}^{(\mu)}(u,v;\alpha,D)\psi^k,\quad(a_k,\mu\neq0;\ \nu,\eta\in\mathbb{N}_0) \tag{4.7}$$

and

$$\begin{aligned}&\Theta_{n,s,\nu,\eta}^m(x,y,z;u,v;\xi)\\ &\quad:=\textstyle\sum_{k=0}^{[\frac{n}{s}]}\frac{a_k}{(n-sk)!}{}_pg_{n-sk}^m(x,y,z;A,B)C_{\nu+\eta k}^{(\mu)}(u,v;\alpha,D)\xi^k,\quad(n,s\in\mathbb{N}).\end{aligned} \tag{4.8}$$

Then we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\Theta_{n,s,\nu,\eta}^m\left(x,y,z,u,v;\frac{\gamma}{t^s}\right)t^n\\ &\qquad=\exp(x\sqrt{2A}\ t)\Phi(y,\sqrt{2A}\ t)\exp(Bzt^m)\Lambda_{\nu,\eta}(u,v;\gamma),\end{aligned} \tag{4.9}$$

provided that each member of (4.9) exists.

Remark. Using equation (4.7) and (4.8) in (4.9) (for $a_k = 1$, $\nu = 0$ and $\eta = 1$), we get

$$\sum_{k=0}^{[\frac{n}{s}]} {}_p g^m_{n-sk}(x, y, z; A, B) C^{(\mu)}_k(u, v; \alpha, D)\gamma^k \frac{t^{n-sk}}{(n-sk)!}$$
$$= \exp(x\sqrt{2A}\ t)\Phi(y, \sqrt{2A}\ t) \exp(Bzt^m) \textstyle\sum_{k=0}^{\infty} C^{(\mu)}_k(u, v; \alpha, D)\gamma^k, \quad (4.10)$$

which on using generating matrix function (4.6) gives

$$\sum_{k=0}^{[\frac{n}{s}]} {}_p g^m_{n-sk}(x, y, z; A, B) C^{(\mu)}_k(u, v; \alpha, D)\gamma^k \frac{t^{n-sk}}{(n-sk)!}$$
$$= \exp(x\sqrt{2A}\ t)\Phi(y, \sqrt{2A}\ t) \exp(Bzt^m)(\alpha I - u\gamma\sqrt{2D} + v\gamma^2 I)^{-\mu}. \quad (4.11)$$

Making appropriate choices of $\Phi(y, t)$ (Table 1) for different 2VgP in equation (4.11), we get bilateral generating matrix function involving the corresponding members of 3VgGHMaP and 2V1PGeMaP, given in Table 14.

Table 14. Bilateral generating matrix function for some members of 3VgGHMaP ${}_p g^m_n(x, y, z; A, B)$ and 2V1PGeMaP $C^{(\mu)}_n(x, y; \alpha, A)$.

S.No.	Values of $\Phi(y, t)$ and Corresponding 2VgP	Bilateral generating matrix function
I.	$\Phi(y, t) = e^{yt^i}$; GHP $H^{(i)}_n(x, y)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{H^{(i)}} g^m_{n-sk}(x, y, z; A, B) C^{(\mu)}_k(u, v; \alpha, D)\gamma^k \frac{t^{n-sk}}{(n-sk)!}$ $= \exp(x\sqrt{2A}\ t + y(\sqrt{2A}\ t)^i) \exp(Bzt^m)(\alpha I - u\gamma\sqrt{2D} + v\gamma^2 I)^{-\mu}$
II.	$\Phi(y, t) = C_0(-yt^i)$; 2VGLP ${}_i L_n(y, x)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{iL} g^m_{n-sk}(x, y, z; A, B) C^{(\mu)}_k(u, v; \alpha, D)\gamma^k \frac{t^{n-sk}}{(n-sk)!}$ $= \exp(x\sqrt{2A}\ t) C_0(-y(\sqrt{2A}\ t)^i) \exp(Bzt^m)(\alpha I - u\gamma\sqrt{2D} + v\gamma^2 I)^{-\mu}$
III.	$\Phi(y, t) = \frac{1}{1-yt^i}$; 2VTEP $e^{(i)}_n(x, y)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{e^{(i)}} g^m_{n-sk}(x, y, z; A, B) C^{(\mu)}_k(u, v; \alpha, D)\gamma^k \frac{t^{n-sk}}{(n-sk)!}$ $= \frac{1}{1-y(\sqrt{2A}\ t)^i} \exp(x\sqrt{2A}\ t) \exp(Bzt^m)(\alpha I - u\gamma\sqrt{2D} + v\gamma^2 I)^{-\mu}$
IV.	$\Phi(y, t) = \frac{t}{\exp(t)-1} \exp(yt^j)$; 2DBP $B^{(j)}_n(x, y)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{B^{(j)}} g^m_{n-sk}(x, y, z; A, B) C^{(\mu)}_k(u, v; \alpha, D)\gamma^k \frac{t^{n-sk}}{(n-sk)!}$ $= \frac{\sqrt{2A}\ t}{\exp(\sqrt{2A}\ t)-1} \exp(x\sqrt{2A}\ t + y(\sqrt{2A}\ t)^j) \exp(Bzt^m)$ $(\alpha I - u\gamma\sqrt{2D} + v\gamma^2 I)^{-\mu}$
V.	$\Phi(y, t) = \frac{2}{\exp(t)+1} \exp(yt^j)$; 2DEP $E^{(j)}_n(x, y)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{E^{(j)}} g^m_{n-sk}(x, y, z; A, B) C^{(\mu)}_k(u, v; \alpha, D)\gamma^k \frac{t^{n-sk}}{(n-sk)!}$ $= \frac{2}{\exp(\sqrt{2A}\ t)+1} \exp(x\sqrt{2A}\ t + y(\sqrt{2A}\ t)^j) \exp(Bzt^m)$ $(\alpha I - u\gamma\sqrt{2D} + v\gamma^2 I)^{-\mu}$

Next, setting $r = 3$ and $\Omega_{\nu+\eta k}(u, v, w) = H^{(r,t)}_n(u, v, w; C)$ in Theorem 4.1, where $H^{(r,t)}_n(u, v, w; C)$ denotes the 3-index 3-variable Hermite matrix polynomials (3I3VHMaP) generated by [10]

$$\exp\left(xt\sqrt{mA} - yt^m I + zt^s I\right) = \sum_{n=0}^{\infty} H^{(m,s)}_n(x, y, z; A)\frac{t^n}{n!}, \quad (4.12)$$

where I is a unit matrix and both I and A are matrices in $\mathbb{C}^{N\times N}$ satisfying condition (1.13). Then, we obtain the following class of bilateral generating matrix relation for 3VgGHMaP ${}_p g^m_n(x, y, z; A, B)$ and 3I3VHMaP $H^{(r,t)}_n(u, v, w; C)$.

Corollary 4.3. *Let*

$$\Lambda_{\nu,\eta}(u,v,w;\psi) := \sum_{k=0}^{\infty} a_k H^{(r,t)}_{\nu+\eta k}(u,v,w;C)\psi^k, \quad (a_k \neq 0;\ \nu,\eta \in \mathbb{N}_0) \tag{4.13}$$

and

$$\Theta^m_{n,s,\nu,\eta}(x,y,z;u,v,w;\xi) := \sum_{k=0}^{[\frac{n}{s}]} \frac{a_k}{(n-sk)!}\, {}_pg^m_{n-sk}(x,y,z;A,B) H^{(r,t)}_{\nu+\eta k}(u,v,w;C)\xi^k, \quad (n,s \in \mathbb{N}). \tag{4.14}$$

Then we have

$$\sum_{n=0}^{\infty} \Theta^m_{n,s,\nu,\eta}\left(x,y,z,u,v,w;\frac{\gamma}{t^s}\right) t^n = \exp(x\sqrt{2A}\ t)\Phi(y,\sqrt{2A}\ t)\exp(Bzt^m)\Lambda_{\nu,\eta}(u,v,w;\gamma), \tag{4.15}$$

provided that each member of (4.15) exists.

Remark. Using equation (4.13) and (4.14) in (4.15) (for $a_k = \frac{1}{k!}$, $\nu = 0$ and $\eta = 1$), we get

$$\sum_{k=0}^{[\frac{n}{s}]} {}_pg^m_{n-sk}(x,y,z;A,B) H^{(r,t)}_k(u,v,w;C)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!} = \exp(x\sqrt{2A}\ t)\Phi(y,\sqrt{2A}\ t)\exp(Bzt^m)\sum_{k=0}^{\infty} H^{(r,t)}_k(u,v,w;C)\frac{\gamma^k}{k!}, \tag{4.16}$$

which on using generating matrix function (4.12) gives

$$\sum_{k=0}^{[\frac{n}{s}]} {}_pg^m_{n-sk}(x,y,z;A,B) H^{(r,t)}_k(u,v,w;C)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!} = \exp(x\sqrt{2A}\ t)\Phi(y,\sqrt{2A}\ t)\exp(Bzt^m)\exp(u\gamma\sqrt{rC} - v\gamma^r I + w\gamma^t I). \tag{4.17}$$

Making appropriate choices of $\Phi(y,t)$ (Table 1) for different 2VgP in equation (4.17), we get bilateral generating matrix function involving the corresponding members of 3VgGHMaP and 3I3VHMaP, given in Table 15.

Table 15. Bilateral generating matrix function for some members of 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$ and 3I3VHMaP $H_n^{(r,t)}(u,v,w;C)$.

S.No.	Values of $\Phi(y,t)$ and Corresponding 2VgP	Bilateral generating matrix function
I.	$\Phi(y,t)=e^{yt^i}$; GHP $H_n^{(i)}(x,y)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{H^{(i)}}g_{n-sk}^m(x,y,z;A,B)H_k^{(r,t)}(u,v,w;C)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!}$ $=\exp(x\sqrt{2A}\ t+y(\sqrt{2A}\ t)^i)\exp(Bzt^m)\exp(u\gamma\sqrt{rC}-v\gamma^r I+w\gamma^t I)$
II.	$\Phi(y,t)=C_0(-yt^i)$; 2VGLP ${}_iL_n(y,x)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{iL}g_{n-sk}^m(x,y,z;A,B)H_k^{(r,t)}(u,v,w;C)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!}$ $=\exp(x\sqrt{2A}\ t)C_0(-y(\sqrt{2A}\ t)^i)\exp(Bzt^m)\exp(u\gamma\sqrt{rC}-v\gamma^r I+w\gamma^t I)$
III.	$\Phi(y,t)=\frac{1}{1-yt^i}$; 2VTEP $e_n^{(i)}(x,y)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{e^{(i)}}g_{n-sk}^m(x,y,z;A,B)H_k^{(r,t)}(u,v,w;C)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!}$ $=\frac{1}{1-y(\sqrt{2A}\ t)^i}\exp(x\sqrt{2A}\ t)\exp(Bzt^m)\exp(u\gamma\sqrt{rC}-v\gamma^r I+w\gamma^t I)$
IV.	$\Phi(y,t)=\frac{t}{\exp(t)-1}\exp(yt^j)$; 2DBP $B_n^{(j)}(x,y)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{B^{(j)}}g_{n-sk}^m(x,y,z;A,B)H_k^{(r,t)}(u,v,w;C)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!}$ $=\frac{\sqrt{2A}\ t}{\exp(\sqrt{2A}\ t)-1}\exp(x\sqrt{2A}\ t+y(\sqrt{2A}\ t)^j)\exp(Bzt^m)$ $\exp(u\gamma\sqrt{rC}-v\gamma^r I+w\gamma^t I)$
V.	$\Phi(y,t)=\frac{2}{\exp(t)+1}\exp(yt^j)$; 2DEP $E_n^{(j)}(x,y)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{E^{(j)}}g_{n-sk}^m(x,y,z;A,B)H_k^{(r,t)}(u,v,w;C)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!}$ $=\frac{2}{\exp(\sqrt{2A}\ t)+1}\exp(x\sqrt{2A}\ t+y(\sqrt{2A}\ t)^j)\exp(Bzt^m)$ $\exp(u\gamma\sqrt{rC}-v\gamma^r I+w\gamma^t I)$

Again in Theorem 4.1, setting $r=3$ and $\Omega_{\nu+\eta k}(u,v,w)={}_qg_{\nu+\eta k}^l(u,v,w;C,D)$ defined by generating function (2.1), we obtain the following result which provides a class of bilinear generating matrix relations for 3VgGHMaP ${}_pg_n^m(x,y,z;A,B)$.

Corollary 4.4. *Let*

$$\Lambda_{\nu,\eta}(u,v,w;\psi):=\sum_{k=0}^{\infty}a_k\,{}_qg_{\nu+\eta k}^l(u,v,w;C,D)\psi^k,\quad (a_k\neq 0;\ \nu,\eta\in\mathbb{N}_0)\tag{4.18}$$

and

$$\Theta_{n,s,\nu,\eta}^m(x,y,z;u,v,w;\xi)$$
$$:=\sum_{k=0}^{[\frac{n}{s}]}\frac{a_k}{(n-sk)!}\,{}_pg_{n-sk}^m(x,y,z;A,B)\,{}_qg_{\nu+\eta k}^l(u,v,w;C,D)\xi^k,\quad (n,p\in\mathbb{N}).\tag{4.19}$$

Then we have

$$\sum_{n=0}^{\infty}\Theta_{n,s,\nu,\eta}^m\left(x,y,z,u,v,w;\frac{\gamma}{t^s}\right)t^n\tag{4.20}$$
$$=\exp(x\sqrt{2A}\ t)\Phi(y,\sqrt{2A}\ t)\exp(Bzt^m)\Lambda_{\nu,\eta}(u,v,w;\gamma),$$

provided that each member of (4.20) exists.

Remark. Using equation (4.18) and (4.19) in (4.20) (for $a_k=\frac{1}{k!}$, $\nu=0$ and $\eta=1$), we get

$$\sum_{k=0}^{[\frac{n}{s}]}{}_pg_{n-sk}^m(x,y,z;A,B)\,{}_qg_k^l(u,v,w;C,D)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!}$$
$$=\exp(x\sqrt{2A}\ t)\Phi(y,\sqrt{2A}\ t)\exp(Bzt^m)\sum_{k=0}^{\infty}{}_qg_k^l(u,v,w;C,D)\frac{\gamma^k}{k!},\tag{4.21}$$

which on using generating matrix function (2.1) gives

$$\sum_{k=0}^{[\frac{n}{s}]} {}_pg^m_{n-sk}(x,y,z;A,B)\,{}_qg^l_k(u,v,w;C,D)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!} = \exp(x\sqrt{2A}\ t)$$
$$\Phi(y,\sqrt{2A}\ t)\exp(Bzt^m)\exp(u\sqrt{2C}\ \gamma)\Phi(v,\sqrt{2C}\ \gamma)\exp(Dw\gamma^l). \qquad (4.22)$$

Making appropriate choices of $\Phi(y,t)$ (Table 1) for different 2VgP in equation (4.22), we get bilinear generating matrix function involving the corresponding members of 3VgGHMaP, given in Table 16.

Table 16. Bilinear generating matrix function for some members of 3VgGHMaP ${}_pg^m_n(x,y,z;A,B)$.

S.No.	Values of $\Phi(y,t)$ and Corresponding 2VgP	Bilinear generating matrix function
I.	$\Phi(y,t)=e^{yt^i}$; GHP $H^{(i)}_n(x,y)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{H^{(i)}}g^m_{n-sk}(x,y,z;A,B)\,{}_{H^{(j)}}g^l_k(u,v,w;C,D)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!} = \exp(x\sqrt{2A}\ t+y(\sqrt{2A}\ t)^i)\exp(Bzt^m)\exp(u\sqrt{2C}\ \gamma+v(\sqrt{2C}\ \gamma)^j)\exp(Dw\gamma^l)$
II.	$\Phi(y,t)=C_0(-yt^i)$; 2VGLP ${}_iL_n(y,x)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{iL}g^m_{n-sk}(x,y,z;A,B)\,{}_{jL}g^l_k(u,v,w;C,D)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!} = \exp(x\sqrt{2A}\ t)C_0(-y(\sqrt{2A}\ t)^i)\exp(Bzt^m)\exp(u\sqrt{2C}\ \gamma)C_0(-v(\sqrt{2C}\ \gamma)^j)\exp(Dw\gamma^l)$
III.	$\Phi(y,t)=\frac{1}{1-yt^i}$; 2VTEP $e^{(i)}_n(x,y)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{e^{(i)}}g^m_{n-sk}(x,y,z;A,B)\,{}_{e^{(j)}}g^l_k(u,v,w;C,D)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!} = \frac{\exp(x\sqrt{2A}\ t)\exp(Bzt^m)\exp(u\sqrt{2C}\ \gamma)\exp(Dw\gamma^l)}{(1-y(\sqrt{2A}\ t)^i)(1-v(\sqrt{2C}\ t)^j)}$
IV.	$\Phi(y,t)=\frac{t}{\exp(t)-1}\exp(yt^i)$; 2DBP $B^{(i)}_n(x,y)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{B^{(i)}}g^m_{n-sk}(x,y,z;A,B)\,{}_{B^{(j)}}g^l_k(u,v,w;C,D)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!} = \frac{2\sqrt{AC}\ t}{(y(\sqrt{2A}\ t)-1)((\sqrt{2C}\ t)-1)}\gamma\exp(x\sqrt{2A}\ t+y(\sqrt{2A}\ t)^i)\exp(Bzt^m)\exp(u\sqrt{2C}\ \gamma+y(\sqrt{2C}\ \gamma)^j)\exp(Dw\gamma^l)$
V.	$\Phi(y,t)=\frac{2}{\exp(t)+1}\exp(yt^i)$; 2DEP $E^{(i)}_n(x,y)$	$\sum_{k=0}^{[\frac{n}{s}]} {}_{E^{(i)}}g^m_{n-sk}(x,y,z;A,B)\,{}_{E^{(j)}}g^l_k(u,v,w;C,D)\frac{\gamma^k}{k!}\frac{t^{n-sk}}{(n-sk)!} = \frac{4\exp(x\sqrt{2A}\ t+y(\sqrt{2A}\ t)^i)}{(\exp(\sqrt{2A}\ t)+1)(\exp(\sqrt{2C}\ t)+1)}\exp(Bzt^m)\exp(u\sqrt{2C}\ \gamma+v(\sqrt{2C}\ \gamma)^j)\exp(Dw\gamma^l)$

5. Conclusion

In this article, the mixed family of special polynomials associated with GHMaP are introduced by means of generating functions and series expansions. These polynomials are framed within the context of monomiality principle and their properties are derived. The family of 3VgGHMaP are introduced and are characterized by their properties. Some bilateral and bilinear generating matrix functions are also established for this hybrid class of matrix polynomials.

Acknowledgments. The authors are thankful to the anonymous referee for several useful comments and suggestions towards the improvement of this paper.

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Ghazala Yasmin
Department of Applied Mathematics, Faculty of Engineering and Technology,
Aligarh Muslim University, Aligarh-202002, India
E-mail address: `ghazala30@gmail.com`

Hibah Islahi
Department of Applied Mathematics, Faculty of Engineering and Technology,
Aligarh Muslim University, Aligarh-202002, India
E-mail address: `islahi.hibah@gmail.com`

Serkan Araci
Department of Economics, Faculty of Economics, Administrative and Social Science,
Hasan Kalyoncu University, TR-27410 Gaziantep, Turkey
E-mail address: `mtsrkn@hotmail.com`