Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 11 Issue 1(2020), Pages 24-42.

GENERALIZED HERMITE-HADAMARD TYPE INEQUALITIES FOR QUASI-CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, we present two new general integral identities for twice differentiable functions. Under the utility of these identities, we establish some new inequalities for classical integrals and Riemann-Liouville fractional integrals of the Hermite-Hadamard's type via functions whose second derivatives absolute values are quasi-convex. The results presented here would provide generalizations of those given in earlier works. Also, we extend our results to function of several variables. At the end, we present applications for means of real numbers and several error approximations for the trapezoidal formula.

1. Introduction and Preliminaries

We begin this section with the following well-known definitions of convexity of functions.

Definition 1.1. *A real-valued function $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on I if the inequality*

$$\phi(tz + (1-t)w) \leq t\phi(z) + (1-t)\phi(w) \tag{1.1}$$

holds for all $z, w \in I$ and $t \in [0,1]$. ϕ is said to be concave on I if the inequality given in (1.1) holds in the reverse direction.

Definition 1.2. *A real-valued function $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex on I if the inequality*

$$\phi(tz + (1-t)w) \leq \max\{\phi(z), \phi(w)\} \tag{1.2}$$

holds for all $z, w \in I$ and $t \in [0,1]$. ϕ is said to be quasi-concave on I if the inequality given in (1.2) holds in the reverse direction.

Every convex function is quasi-convex, but the converse is not generally true (see for instance [10–13]), so quasi convexity is the generalization of convexity of a function.

2000 *Mathematics Subject Classification.* 26D15, 26A51, 26D20.
Key words and phrases. quasi-convex function, Hermite-Hadamard inequality, fractional integrals, trapezoidal formula, error approximation.
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Submitted October 15, 2019. February 5, 2020.
Communicated by Guest editor Artion Kashuri.
The research was supported by the Natural Science Foundation of China (Grant nos. 61673169, 11701176, 11626101, and 11601485),

A number of important inequalities have been obtained for the class of convex functions, when the idea of convexity was introduced more than a hundred years ago. But among those one of the most prominent inequality is Hermite-Hadamard's inequality (or Hadamard's inequality), which is presented in the following theorem.

Theorem 1.3. *(Hermite-Hadamard's inequality, see [26]) If the function $\phi : I \to \mathbb{R}$ is convex on I such that $c, d \in I$ with $c < d$. Then the double inequality*

$$\phi\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \phi(z)dz \leq \frac{\phi(c)+\phi(d)}{2} \tag{1.3}$$

holds. If the function ϕ is concave on I, then inequality (1.3) holds in the reverse order.

Hermite-Hadamard's inequality gives an estimate from both sides of the mean, i.e. from above and below of the mean value of a convex function and ensures the integrability of any convex function too. It is also a matter of great interest and one has to note that some of the classical important inequalities for means can be obtained from Hadamard's inequality under the utility of particular convex functions ϕ. These inequalities for convex functions play a crucial role in analysis as well as in other areas of pure and applied mathematics.

Barani et al. [15] have obtained the following inequalities (which are the refinements of the results given in [10]) for functions whose second derivatives absolute values are quasi-convex.

Theorem 1.4 ([15])**.** *Let $\phi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $\phi'' \in L[c,d]$, $c, d \in I^\circ$ with $c < d$. If $|\phi''|$ is quasi-convex on $[c,d]$, then we have*

$$\begin{aligned} &\left|\frac{\phi(c)+\phi(d)}{2} - \frac{1}{d-c}\int_c^d \phi(z)dz\right| \\ &\leq \frac{(d-c)^2}{24}\left[\max\left\{|\phi''(c)|, \left|\phi''\left(\frac{c+d}{2}\right)\right|\right\} + \max\left\{|\phi''(d)|, \left|\phi''\left(\frac{c+d}{2}\right)\right|\right\}\right], \end{aligned} \tag{1.4}$$

where and in what follows I° denotes the interior of I.

Theorem 1.5 ([15])**.** *Let $\phi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $\phi'' \in L[c,d]$, $c, d \in I^\circ$ with $c < d$. If $p, q > 1$ such that $p^{-1} + q^{-1} = 1$ and $|\phi''|^q$ is quasi-convex on $[c,d]$, then one has*

$$\begin{aligned} \left|\frac{\phi(c)+\phi(d)}{2} - \frac{1}{d-c}\int_c^d \phi(z)dz\right| &\leq M^{1/p}\frac{(d-c)^2}{16}\left[\left(\max\left\{|\phi''(c)|^q, \left|\phi''\left(\frac{c+d}{2}\right)\right|^q\right\}\right)^{1/q}\right. \\ &\left. + \left(\max\left\{|\phi''(c)|^q, \left|\phi''\left(\frac{c+d}{2}\right)\right|^q\right\}\right)^{1/q}\right], \end{aligned} \tag{1.5}$$

where $M = \Gamma(1/2)\Gamma(p+1)/[2\Gamma(p+3/2)]$ and $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ is the Gamma function.

Theorem 1.6 ([15])**.** *Let $\phi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $\phi'' \in L[c,d]$, $c, d \in I^\circ$ with $c < d$. If $|\phi''|^q$ is quasi-convex on $[c,d]$ for $q \geq 1$,* *then*

$$\left|\frac{\phi(c)+\phi(d)}{2}-\frac{1}{d-c}\int_c^d \phi(z)dz\right| \le \frac{(d-c)^2}{24}\left[\left(\max\left\{|\phi''(c)|^q, \left|\phi''\left(\frac{c+d}{2}\right)\right|^q\right\}\right)^{1/q} + \left(\max\left\{|\phi''(d)|^q, \left|\phi''\left(\frac{c+d}{2}\right)\right|^q\right\}\right)^{1/q}\right]. \tag{1.6}$$

Before writing Hermite-Hadamard inequality for fractional integrals, we first recall the definition of fractional integrals [25,37].

Definition 1.7. *Let $\phi \in L[c,d]$ with $c \ge 0$. Then the left-sided and right-sided Riemann-Liouville fractional integrals $J^{\eta}_{c+}\phi$ and $J^{\eta}_{d-}\phi$ of order $\eta > 0$ are defined by*

$$J^{\eta}_{c+}\phi(z) = \frac{1}{\Gamma(\eta)}\int_c^z (z-s)^{\eta-1}\phi(s)ds \qquad (z>c)$$

and

$$J^{\eta}_{d-}\phi(z) = \frac{1}{\Gamma(\eta)}\int_z^d (s-z)^{\eta-1}\phi(s)ds, \qquad (z<d)$$

respectively. Here, $\Gamma(\eta)$ is the Gamma function given by

$$\Gamma(\eta) = \int_0^\infty e^{-u}u^{\eta-1}du. \tag{1.7}$$

It is also important to note that $J^0_{c+}\phi(z) = J^0_{d-}\phi(z) = \phi(z)$ and the fractional integral shrinks to the classical integral in the case of $\eta = 1$.

Sarikaya et al. [38] established the Hermite-Hadamard's inequality for fractional integrals as follows.

Theorem 1.8 ([38])**.** *Let $0 \le c < b$ and $\phi : [c,d] \to \mathbb{R}^+$ be a function such that $\phi \in L[c,d]$. If ϕ is convex on $[c,d]$, then the following double inequality holds*

$$\phi\left(\frac{c+d}{2}\right) \le \frac{\Gamma(\eta+1)}{2(d-c)^{\eta}}\left[J^{\eta}_{c+}\phi(d) + J^{\eta}_{d-}\phi(c)\right] \le \frac{\phi(c)+\phi(d)}{2}. \tag{1.8}$$

For more recent results, extensions, generalizations and refinements concerning to Hermite-Hadamard inequality see [1–9,14,16–24,26–36,39–43] and the references cited over there.

The main purpose of this paper is to establish generalized inequalities for right hand side of Hermite-Hadamard inequality via classical and fractional integrals using a function whose second derivative absolute values are quasi convex. Also we relate our results to the results obtained by Barani et al. [15]. Then we extend our results to functions of several variables. At the end, we give some applications for means of real numbers and for trapezoidal formula.

2. Hermite type inequalities for quasi-convex function via classical integrals

In the beginning of this section, for the sake of simplification we first introduce the function $\Upsilon(z)$.

Let $\phi : I \to \mathbb{R}$ be twice differentiable on I° and $c, d \in I^\circ$ with $c < d$. If $\phi'' \in L[c,d]$, then for all $z \in [c,d]$, we define

$$\Upsilon(z) = \frac{\phi'(z)\left((z-c)^2 - (d-z)^2\right) + 2\phi(c)(z-c) + 2\phi(d)(d-z)}{2(d-c)}. \tag{2.1}$$

Letting $z = \frac{c+d}{2}$ in (2.1), then we have

$$\Upsilon\left(\frac{c+d}{2}\right) = \frac{\phi(c)+\phi(d)}{2}.$$

Lemma 2.1. *Let $\phi : I \to \mathbb{R}$ be twice differentiable on I° and $c, d \in I^\circ$ with $c < d$. If $\phi'' \in L[c,d]$, then the identity*

$$\begin{aligned}\Upsilon(z) - \frac{1}{d-c}\int_c^d \phi(u)du &= \frac{(z-c)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)\phi''(sc+(1-s)z)ds \\ &+ \frac{(d-z)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)\phi''(sd+(1-s)z)ds. \end{aligned}\tag{2.2}$$

is valid for all $z \in [c,d]$.

Proof. By the well-known property of integration (i.e. Integration by parts) and then by changing of variables, we have

$$\begin{aligned}&\frac{(z-c)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)\phi''(sc+(1-s)z)ds + \frac{(d-z)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)\phi''(sd+(1-s)z)ds \\ &= \frac{(z-c)^3}{2(d-c)}\left[\frac{-\phi'(z)}{c-z} + 2\left(\frac{\phi(c)}{(c-z)^2} - \frac{1}{(c-z)^3}\int_z^c \phi(u)du\right)\right] \\ &+ \frac{(d-z)^3}{2(d-c)}\left[\frac{-\phi'(z)}{d-z} + 2\left(\frac{\phi(d)}{(d-z)^2} - \frac{1}{(d-z)^3}\int_z^d \phi(u)du\right)\right],\end{aligned}$$

and so the left hand side of (2.2) is obvious. □

Theorem 2.2. *Let all the requisites of Lemma 2.1 hold. Additionally, if $|\phi''|$ is quasi-convex on $[c,d]$, then the inequality*

$$\begin{aligned}&\left|\Upsilon(z) - \frac{1}{d-c}\int_c^d \phi(u)du\right| \\ &\leq \frac{\max\{|\phi''(c)|, |\phi''(z)|\}(z-c)^3 + \max\{|\phi''(d)|, |\phi''(z)|\}(d-z)^3}{3(d-c)}\end{aligned}\tag{2.3}$$

holds for all $z \in [c,d]$.

Proof. It follows from the identity (2.2) and triangle inequality of real numbers together with the quasi-convexity of $|\phi''|$ that

$$\begin{aligned}\left|\Upsilon(z)-\frac{1}{d-c}\int_c^d\phi(u)du\right| &\leq \frac{(z-c)^3}{2(d-c)}\int_0^1\left(1-s^2\right)|\phi''(sc+(1-s)z)|ds\\ &\quad+\frac{(d-z)^3}{2(d-c)}\int_0^1\left(1-s^2\right)|\phi''(sd+(1-s)z)|ds\\ &\leq \frac{(z-c)^3}{2(d-c)}\int_0^1\left(1-s^2\right)\max\{|\phi''(c)|,|\phi''(z)|\}ds\\ &\quad+\frac{(d-z)^3}{2(d-c)}\int_0^1\left(1-s^2\right)\max\{|\phi''(d)|,|\phi''(z)|\}ds\\ &=\frac{\max\{|\phi''(c)|,|\phi''(z)|\}(z-c)^3+\max\{|\phi''(d)|,|\phi''(z)|\}(d-z)^3}{3(d-c)}.\end{aligned}$$

The proof is completed. □

Remark. *By putting $z=\frac{c+d}{2}$ in the inequality (2.3), we have*

$$\begin{aligned}&\left|\frac{\phi(c)+\phi(d)}{2}-\frac{1}{d-c}\int_c^d\phi(z)dz\right|\\ &\leq\frac{(d-c)^2}{24}\left[\max\left\{|\phi''(c)|,\left|\phi''\left(\frac{c+d}{2}\right)\right|\right\}+\max\left\{|\phi''(d)|,\left|\phi''\left(\frac{c+d}{2}\right)\right|\right\}\right],\end{aligned}$$

which is the inequality (1.4).

Theorem 2.3. *Let all the requisites of Lemma 2.1 hold. Additionally, if $|\phi''|^q$ is quasi-convex on $[c,d]$ for $q>1$ and $p^{-1}=1-q^{-1}$. Then the inequality*

$$\begin{aligned}&\left|\Upsilon(z)-\frac{1}{d-c}\int_c^d\phi(u)du\right|\\ &\leq M^{1/p}\frac{\left(\max\{|\phi''(c)|^q,|\phi''(z)|^q\}\right)^{1/q}(z-c)^3+\left(\max\{|\phi''(d)|^q,|\phi''(z)|^q\}\right)^{1/q}(d-z)^3}{2(d-c)}\end{aligned}\tag{2.4}$$

holds for all $z\in[c,d]$, where $M=\Gamma(1/2)\Gamma(p+1)[2\Gamma(p+3/2)]$.

Proof. Under the utility of Lemma 2.1, using triangle and Hölder inequalities, we have

$$\begin{aligned}&\left|\Upsilon(z)-\frac{1}{d-c}\int_c^d\phi(u)du\right|\\ &\leq\frac{(z-c)^3}{2(d-c)}\left(\int_0^1\left(1-s^2\right)^p ds\right)^{1/p}\left(\int_0^1|\ \phi''(sc+(1-s)z)\ |^q\ ds\right)^{1/q}\\ &\quad+\frac{(d-z)^3}{2(d-c)}\left(\int_0^1\left(1-s^2\right)^p ds\right)^{1/p}\left(\int_0^1|\ \phi''(sd+(1-s)z)\ |^q\ ds\right)^{1/q}.\end{aligned}\tag{2.5}$$

Using the quasi-convexity of $|\phi''|^q$, we obtain

$$\int_0^1|\phi''(sc+(1-s)z)|^q dt\leq\max\{|\phi''(c)|^q,|\phi''(z)|^q\},\tag{2.6}$$

similarly

$$\int_0^1 |\phi''(sd+(1-s)z)|^q ds \leq \max\{|\phi''(d)|^q, |\phi''(z)|^q\}. \tag{2.7}$$

Also we have

$$\left(\int_0^1 \left(1-s^2\right)^p ds\right)^{1/p} = \left(\frac{1}{2}\int_0^1 (1-z)^p z^{-\frac{1}{2}} dz\right)^{1/p} = \left(\frac{1}{2}\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{1/p} = M^{1/p}. \tag{2.8}$$

Using the inequalities (2.8), (2.7) and (2.6) in the inequality (2.5), we get the desired conclusion. □

Remark. *By putting $z=\frac{c+d}{2}$ in the inequality (2.4), we get the inequality (1.5), that is*

$$\left|\frac{\phi(c)+\phi(d)}{2} - \frac{1}{d-c}\int_c^d \phi(z)dz\right| \leq M^{\frac{1}{p}}\frac{(d-c)^2}{16}\left[\left(\max\left\{|\phi''(c)|^q, \left|\phi''\left(\frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} + \left(\max\left\{|\phi''(c)|^q, \left|\phi''\left(\frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right], \tag{2.9}$$

where $M=\Gamma(1/2)\Gamma(p+1)/[2\Gamma(p+3/2)]$.

Theorem 2.4. *Let all the requisites of Lemma 2.1 hold. Additionally, if $q \geq 1$ and $|\phi''|^q$ is quasi-convex on $[c,d]$, then the inequality*

$$\left|\Upsilon(z) - \frac{1}{d-c}\int_c^d \phi(u)du\right| \leq \frac{(\max\{|\phi''(c)|^q, |\phi''(z)|^q\})^{\frac{1}{q}}(z-c)^3 + (\max\{|\phi''(d)|^q, |\phi''(z)|^q\})^{\frac{1}{q}}(d-z)^3}{3(d-c)} \tag{2.10}$$

holds for all $z \in [c,d]$.

Proof. Emplying Lemma 2.1 and then using the well-known Hölder inequality, we arrive at the following

$$\begin{aligned}
&\left|\Upsilon(z) - \frac{1}{d-c}\int_c^d \phi(u)du\right| \\
&\leq \frac{(z-c)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)|\phi''(sc+(1-s)z)|ds \\
&\quad + \frac{(d-z)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)|\phi''(sd+(1-s)z)|ds \\
&\leq \frac{(z-c)^3}{2(d-c)}\left(\int_0^1 (1-s^2)ds\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(1-s^2\right)|\phi''(sc+(1-s)z)|^q ds\right)^{\frac{1}{q}} \\
&+ \frac{(d-z)^3}{2(d-c)}\left(\int_0^1 (1-s^2)ds\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(1-s^2\right)|\phi''(sd+(1-s)z)|^q ds\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.11}$$

Using the quasi-convexity of $|\phi''|^q$, we obtain

$$\int_0^1 (1-s^2)|\phi''(sc+(1-s)z)|^q dt \leq \frac{2}{3}\max\{|\phi''(c)|^q, |\phi''(z)|^q\}, \tag{2.12}$$

similarly

$$\int_0^1 (1-s^2)|\phi''(sd+(1-s)z)|^q ds \leq \frac{2}{3}\max\{|\phi''(d)|^q, |\phi''(z)|^q\}. \tag{2.13}$$

Also we have

$$\left(\int_0^1 (1-s^2)ds\right)^{1-\frac{1}{q}} = \left(\frac{2}{3}\right)^{1-\frac{1}{q}}. \tag{2.14}$$

Using the inequalities (2.14), (2.13) and (2.12) in the inequality (2.11), we have the required conclusion. □

Remark. *By putting* $z = \frac{c+d}{2}$ *in the inequality (2.10) , we have*

$$\left|\frac{\phi(c)+\phi(d)}{2} - \frac{1}{d-c}\int_c^d \phi(z)dz\right|$$
$$\leq \frac{(d-c)^2}{24}\left[\left(\max\left\{|\phi''(c)|^q, \left|\phi''\left(\frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} + \left(\max\left\{|\phi''(d)|^q, \left|\phi''\left(\frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right],$$

which is the inequality (1.6).

3. Hermite-Hadamard type Inequalities for Quasi-convex Functions via Fractional Integrals

In this section we present new Hermite-Hadamard type inequalities for quasi-convex functions via Fractional Integrals.

Let $\phi : I \to \mathbb{R}$ be twice differentiable on I° and $c, d \in I^\circ$ with $c < d$. If $\phi'' \in L[c,d]$, then for all $z \in [c,d]$ and $\eta > 0$, for simplicity we introduce the function $\tilde{\Upsilon}$ defined by

$$\tilde{\Upsilon}(z) = \frac{\phi'(z)\left((z-c)^{\eta+1} - (d-z)^{\eta+1}\right) + (\eta+1)\phi(c)(z-c)^\eta + (\eta+1)\phi(d)(d-z)^\eta}{(\eta+1)(d-c)}.$$

Lemma 3.1. *Let all the requisites of Lemma 2.1 hold with* $I = [0,\infty)$*. Then the identity*

$$\tilde{\Upsilon}(z) - \frac{\Gamma(\eta+1)}{d-c}\left[J^\eta_{c^+}\phi(z) + J^\eta_{d^-}\phi(z)\right] = \frac{(z-c)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 \left(1-s^{\eta+1}\right)\phi''(sc+(1-s)z)ds$$
$$+ \frac{(d-z)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 \left(1-s^{\eta+1}\right)\phi''(sd+(1-s)z)ds \tag{3.1}$$

is valid for all $z \in [c,d]$ *and* $\eta > 0$.

Proof. By integration by parts and then by changing of variables, we obtain

$$\int_0^1 \left(1-s^{\eta+1}\right)\phi''(sc+(1-s)z)ds = \frac{\phi'(z)}{z-c} + \frac{(\eta+1)\phi(c)}{(c-z)^2} - \frac{\Gamma(\eta+2)}{(z-c)^{\eta+2}}J^\eta_{c^+}\phi(z), \tag{3.2}$$

similarly

$$\int_0^1 \left(1-s^{\eta+1}\right)\phi''(sd+(1-s)z)ds = \frac{-\phi'(z)}{d-z} + \frac{(\eta+1)\phi(d)}{(d-z)^2} - \frac{\Gamma(\eta+2)}{(d-z)^{\eta+2}}J^\eta_{d^-}\phi(z). \tag{3.3}$$

Now multiplying (3.2) by $\frac{(z-c)^{\eta+2}}{(\eta+1)(d-c)}$ and (3.3) by $\frac{(d-z)^{\eta+2}}{(\eta+1)(d-c)}$ and then adding we get the identity (3.1). □

Remark. *Lemma 3.1 shrinks to Lemma 2.1 by setting $\eta = 1$.*

Theorem 3.2. *Let all the requisites of Lemma 3.1 hold. Additionally, if $|\phi''|$ is quasi-convex on $[c, d]$, then the inequality*

$$\left|\tilde{\Upsilon}(z) - \frac{\Gamma(\eta+1)}{d-c}\left[J^{\eta}_{c^+}\phi(z) + J^{\eta}_{d^-}\phi(z)\right]\right|$$
$$\leq \frac{\max\{|\phi''(c)|, |\phi''(z)|\}(z-c)^{\eta+2} + \max\{|\phi''(d)|, |\phi''(z)|\}(d-z)^{\eta+2}}{(\eta+2)(d-c)}. \tag{3.4}$$

holds for all $z \in [c, d]$ and $\eta > 0$.

Proof. Under the utility of Lemma 3.1 and using the quasi-convexity of $|\phi''|$, we get

$$\begin{aligned}
&\left|\tilde{\Upsilon}(z) - \frac{\Gamma(\eta+1)}{(d-c)}J^{\eta}_{c^+}\phi(z) + J^{\eta}_{d^-}\phi(z)\right| \\
&\leq \frac{(z-c)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 (1-s^{\eta+1})|\phi''(sc+(1-s)z)|ds \\
&+\frac{(d-z)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 (1-s^{\eta+1})|\phi''(sd+(1-s)z)|ds \\
&\leq \frac{(z-c)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 \left(1-s^{\eta+1}\right)\max\{|\phi''(c)|, |\phi''(z)|\}ds \\
&+\frac{(d-z)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 \left(1-s^{\eta+1}\right)\max\{|\phi''(d)|, |\phi''(z)|\}ds \\
&= \frac{\max\{|\phi''(c)|, |\phi''(z)|\}(z-c)^{\eta+2} + \max\{|\phi''(d)|, |\phi''(z)|\}(d-z)^{\eta+2}}{(\eta+2)(d-c)}.
\end{aligned}$$

□

Corollary 3.3. *Under the assumptions of Theorem 3.2, one has*

$$\begin{aligned}
&\left|\left(\frac{d-c}{2}\right)^{\eta-1}\frac{\phi(c)+\phi(d)}{2} - \frac{\Gamma(\eta+1)}{(d-c)}\left[J^{\eta}_{c^+}\phi\left(\frac{c+d}{2}\right) + J^{\eta}_{d^-}\phi\left(\frac{c+d}{2}\right)\right]\right| \\
&\leq \frac{(d-c)^{\eta+1}}{2^{\eta+2}(\eta+2)}\left[\max\left\{|\phi''(c)|, \left|\phi''\left(\frac{c+d}{2}\right)\right|\right\} + \max\left\{|\phi''(d)|, \left|\phi''\left(\frac{c+d}{2}\right)\right|\right\}\right] \\
&\leq \left(\frac{d-c}{2}\right)^{\eta+1}\frac{1}{\eta+2}\max\left\{|\phi''(c)|, |\phi''(d)|\right\}.
\end{aligned} \tag{3.5}$$

Proof. The first inequality in (3.5) is obtained by choosing $z = \frac{c+d}{2}$ in the inequality (3.4), while the second inequality can be obtained by using quasi convexity of $|\phi''|$. □

Remark. *By putting $\eta = 1$ in Theorem 3.2, we obtain the inequality given in Theorem 2.2.*

Theorem 3.4. *Let all the requisites of Lemma 3.1 hold. Additionally, if $|\phi''|^q$ is quasi-convex on $[c,d]$ for $q>1$ and $p^{-1}=1-q^{-1}$. Then for all $z\in[c,d]$, the inequality given below is true*

$$\left|\tilde{\Upsilon}(z)-\frac{\Gamma(\eta+1)}{d-c}\left[J^{\eta}_{c^+}\phi(z)+J^{\eta}_{d^-}\phi(z)\right]\right|$$
$$\leq M^{\frac{1}{p}}\frac{(\max\{|\phi''(c)|^q,|\phi''(z)|^q\})^{\frac{1}{q}}(z-c)^{\eta+2}+(\max\{|\phi''(d)|^q,|\phi''(z)|^q\})^{\frac{1}{q}}(d-z)^{\eta+2}}{(\eta+1)(d-c)},\tag{3.6}$$

where $M=\Gamma(1+p)\Gamma(1/(1+\eta))/[(\eta+1)\Gamma(1+p+1/(\eta+1))]$.

Proof. Under the utility of Lemma 3.1, using triangle and Hölder inequalities, we have

$$\left|\tilde{\Upsilon}(z)-\frac{\Gamma(\eta+1)}{d-c}\left[J^{\eta}_{c^+}\phi(z)+J^{\eta}_{d^-}\phi(z)\right]\right|$$
$$\leq\frac{(z-c)^{\eta+2}}{(\eta+1)(d-c)}\left(\int_0^1\left(1-s^{\eta+1}\right)^p ds\right)^{\frac{1}{p}}\left(\int_0^1|\,\phi''(sc+(1-s)z)\,|^q\,ds\right)^{\frac{1}{q}}$$
$$+\frac{(d-z)^{\eta+2}}{(\eta+1)(d-c)}\left(\int_0^1\left(1-s^{\eta+1}\right)^p ds\right)^{\frac{1}{p}}\left(\int_0^1|\,\phi''(sd+(1-s)z)\,|^q\,ds\right)^{\frac{1}{q}}.\tag{3.7}$$

Using the quasi-convexity of $|\phi''|^q$, we obtain

$$\int_0^1|\phi''(sc+(1-s)z)|^q dt\leq\max\{|\phi''(c)|^q,|\phi''(z)|^q\},\tag{3.8}$$

similarly

$$\int_0^1|\phi''(sd+(1-s)z)|^q ds\leq\max\{|\phi''(d)|^q,|\phi''(z)|^q\}.\tag{3.9}$$

Also we have

$$\int_0^1\left(1-s^{\eta+1}\right)^p ds=\frac{\int_0^1(1-s)^p s^{\frac{1}{\eta+1}-1}ds}{\eta+1}=\frac{\Gamma(1+p)\Gamma(\frac{1}{\eta+1})}{(\eta+1)\Gamma(1+p+\frac{1}{\eta+1})}=M.\tag{3.10}$$

Using the inequalities (3.10), (3.9) and (3.8) in the inequality (3.7), we have the required conclusion. □

Corollary 3.5. *Under the assumptions of Theorem 2.3, we have*

$$\left|\left(\frac{d-c}{2}\right)^{\eta-1}\frac{\phi(c)+\phi(d)}{2}-\frac{\Gamma(\eta+1)}{(d-c)}\left[J^{\eta}_{c^+}\phi\left(\frac{c+d}{2}\right)+J^{\eta}_{d^-}\phi\left(\frac{c+d}{2}\right)\right]\right|$$
$$\leq M^{\frac{1}{p}}\frac{(d-c)^{\eta+1}}{2^{\eta+2}(\eta+1)}\left[\left(\max\left\{|\phi''(c)|^q,\left|\phi''\left(\frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\max\left\{|\phi''(d)|^q,\left|\phi''\left(\frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right]$$
$$\leq M^{\frac{1}{p}}\frac{(d-c)^{\eta+1}}{2^{\eta+1}(\eta+1)}\left(\max\left\{|\phi''(c)|^q,|\phi''(d)|^q\right\}\right)^{\frac{1}{q}},\tag{3.11}$$

where $M=\Gamma(1+p)\Gamma(1/(1+\eta))/[(\eta+1)\Gamma(1+p+1/(\eta+1))]$.

Proof. The first inequality in (3.11) is obtained by putting $z = \frac{c+d}{2}$ in the inequality (3.6). The second inequality in (3.11) can be obtained by using the quasi-convexity of $|\phi''|^q$. □

Remark. *By putting $\eta = 1$ in Theorem 3.4, we obtain the inequality given in Theorem 2.3.*

Theorem 3.6. *Let all the requisites of Lemma 3.1 hold. Additionally, if for $q \geq 1$ and $|\phi''|^q$ is quasi-convex on $[c, d]$, then for all $z \in [c, d]$, the inequality given below is true*

$$\begin{aligned}
&\left|\tilde{\Upsilon}(z) - \frac{\Gamma(\eta+1)}{d-c}\left[J^{\eta}_{c^+}\phi(z) + J^{\eta}_{d^-}\phi(z)\right]\right| \\
&\leq \frac{\left(\max\{|\phi''(c)|^q, |\phi''(z)|^q\}\right)^{\frac{1}{q}}(z-c)^{\eta+2} + \left(\max\{|\phi''(d)|^q, |\phi''(z)|^q\}\right)^{\frac{1}{q}}(d-z)^{\eta+2}}{(\eta+2)(d-c)} \quad (3.12)
\end{aligned}$$

Proof. Emplying Lemma 3.1 and then using the well-known Hölder inequality, we arrive at the following

$$\begin{aligned}
&\left|\tilde{\Upsilon}(z) - \frac{\Gamma(\eta+1)}{(d-c)} J^{\eta}_{c^+}\phi(z) + J^{\eta}_{d^-}\phi(z)\right| \\
&\leq \frac{(z-c)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 \left(1-s^{\eta+1}\right)|\phi''(sc+(1-s)z)|ds \\
&\quad + \frac{(d-z)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 \left(1-s^{\eta+1}\right)|\phi''(sd+(1-s)z)|ds \\
&\leq \frac{(z-c)^{\eta+2}}{(\eta+1)(d-c)}\left(\int_0^1 (1-s^{\eta+1})ds\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(1-s^{\eta+1}\right)|\phi''(sc+(1-s)z)|^q ds\right)^{\frac{1}{q}} \\
&\quad + \frac{(d-z)^{\eta+2}}{(\eta+1)(d-c)}\left(\int_0^1 (1-s^{\eta+1})ds\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(1-s^{\eta+1}\right)|\phi''(sd+(1-s)z)|^q ds\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.13}$$

Using the quasi-convexity of $|\phi''|^q$, we obtain

$$\int_0^1 (1-s^{\eta+1})|\phi''(sc+(1-s)z)|^q dt \leq \frac{\eta+1}{\eta+2}\max\{|\phi''(c)|^q, |\phi''(z)|^q\}, \tag{3.14}$$

similarly

$$\int_0^1 (1-s^{\eta+1})|\phi''(sd+(1-s)z)|^q ds \leq \frac{\eta+1}{\eta+2}\max\{|\phi''(d)|^q, |\phi''(z)|^q\}. \tag{3.15}$$

Also we have

$$\left(\int_0^1 (1-s^2)ds\right)^{1-\frac{1}{q}} = \left(\frac{\eta+1}{\eta+2}\right)^{1-\frac{1}{q}}. \tag{3.16}$$

Using the inequalities (3.16), (3.15) and (3.14) in the inequality (2.11), we have the required conclusion. □

Corollary 3.7. *Under the assumptions of Theorem 2.4, we have*

$$\begin{aligned}
&\left|\left(\frac{d-c}{2}\right)^{\eta-1}\frac{\phi(c)+\phi(d)}{2}-\frac{\Gamma(\eta+1)}{(d-c)}\left[J^{\eta}_{c^+}\phi\left(\frac{c+d}{2}\right)+J^{\eta}_{d^-}\phi\left(\frac{c+d}{2}\right)\right]\right|\\
&\leq\frac{(d-c)^{\eta+1}}{2^{\eta+2}(\eta+2)}\left[\max\left\{|\phi''(c)|^q,\left|\phi''\left(\frac{c+d}{2}\right)\right|^q\right\}+\max\left\{|\phi''(d)|^q,\left|\phi''\left(\frac{c+d}{2}\right)\right|^q\right\}\right]\\
&\leq\frac{(d-c)^{\eta+1}}{2^{\eta+1}(\eta+2)}\max\left\{|\phi''(c)|^q,|\phi''(d)|^q\right\}. \qquad (3.17)
\end{aligned}$$

Proof. The first inequality in (3.17) is obtained by putting $z=\frac{c+d}{2}$ in the inequality (3.12). The second inequality in (3.17) can be obtained by using the quasi-convexity of $|\phi''|^q$.

Remark. *By putting $\eta = 1$ in Theorem 3.6, we obtain the inequality given in Theorem 2.4.*

□

4. An extension to function of several variables

Let us denote an open and convex subset of $\mathbb{R}^n$, where $n \in N$, by O. We say that a function $\phi : O \to \mathbb{R}$ is quasi-convex on O, if the inequality,

$$\phi(t\alpha+(1-t)\beta)\leq\max\{\phi(\alpha),\phi(\beta)\}$$

holds for all $\alpha, \beta \in O$ and $t \in [0, 1]$.
The following auxiliary result holds true.

Lemma 4.1. *Assume $\phi : O \to \mathbb{R}$ is a function. Then ϕ is quasi-convex on O if and only if for every $\alpha, \beta \in O$, the function $\Delta : [0, 1] \to \mathbb{R}$, $\Delta(t) = \phi((1-t)\alpha + t\beta)$ is quasi-convex on $[0, 1]$.*

Proof. Let $\alpha, \beta \in O$ be fixed. Assume that $\Delta : [0, 1] \to \mathbb{R}$, $\Delta(t) = \phi((1-t)\alpha + t\beta)$ is quasi-convex on $[0, 1]$. Let $\mu \in [0, 1]$ be arbitrary, but fixed, and thus

$$\phi((1-\mu)\alpha+\mu\beta)=\Delta(\mu)=\Delta((1-\mu).0+\mu.1)\leq\max\{\Delta(0),\Delta(1)\}=\max\{\phi(\alpha),\phi(\beta)\}.$$

It follows that ϕ is quasi-convex on O.
Conversely, assume that ϕ is quasi-convex on O. Let $\alpha, \beta \in O$ be fixed. We define Δ, by

$$\Delta(t)=\phi((1-t)\alpha+t\beta),\quad \alpha,\beta\in O,\quad t\in[0,1],$$

we show that Δ is quasi-convex on $[0, 1]$.
Let $\mu_1, \mu_2 \in [0, 1]$ and $\nu \in [0, 1]$. Then

$$\begin{aligned}
\Delta((1-\nu)\mu_1+\nu\mu_2)&=\phi([1-(1-\nu)\mu_1-\nu\mu_2]\alpha+[(1-\nu)\mu_1+\nu\mu_2]\beta)\\
&=\phi((1-\nu)(1-\mu_1)\alpha+\mu_1\beta))+\nu(1-\mu_2)\alpha+\mu_2\beta))\\
&\leq\max\{\phi((1-\mu_1)\alpha+\mu_1\beta),\phi((1-\mu_2)\alpha+\mu_2\beta)\}\\
&=\max\{\Delta(\mu_1),\Delta(\mu_2)\},
\end{aligned}$$

which shows that Δ is quasi-convex on [0, 1].
The proof is completed. □

Proposition 4.2. *Assume $\phi : O \to \mathbb{R}^+$ is a quasi-convex function on O. Then for any $\alpha, \beta \in O$ and any $c, d \in (0,1)$ the following inequality holds true*

$$\begin{aligned}
&\left|\Omega(z) - \frac{1}{d-c}\int_c^d \int_0^u \int_0^y \phi((1-x)\alpha + x\beta)dxdydu\right| \\
&\leq \frac{1}{3(d-c)}\Bigg[(z-c)^3 \max\Big\{|\phi((1-c)\alpha + c\beta))|, |\phi((1-z)\alpha + z\beta)|\Big\} \\
&\quad + (d-z)^3 \max\Big\{|\phi((1-d)\alpha + d\beta)|, |\phi((1-z)\alpha + z\beta)|\Big\}\Bigg], \qquad (4.1)
\end{aligned}$$

where

$$\begin{aligned}
\Omega(z) = &\frac{1}{2(d-c)}\Bigg[\Big((z-c)^2 - (d-z)^2\Big)\int_0^z \Delta(x)dx + 2(d-z)\int_0^d\int_0^y \phi((1-x)\alpha + x\beta)dxdy \\
&+ 2(z-c)\int_0^c\int_0^y \phi((1-x)\alpha + x\beta)dxdy\Bigg],
\end{aligned}$$

for all $z \in [c,d]$, $y \in (0,1)$.

Proof. We fix $\alpha, \beta \in O$ and $c, d \in [0,1]$ with$c < d$. Since ϕ is quasi-convex, by Lemma 4.1 it follows that the function $\Delta : [0,1] \to \mathbb{R}$, defined by, $\Delta(t) = \phi((1-t)\alpha + t\beta)$ is quasi-convex on $[0,1]$.

Let $\Psi : [0,1] \to \mathbb{R}$ be defined by

$$\Psi(t) = \int_0^t\int_0^y \Delta(x)dxdy = \int_0^t\int_0^y \phi((1-x)\alpha + x\beta)dxdy, \quad \forall\ y \in (0,1).$$

Then we clearly see that $\Psi''(t) = \Delta(t)$ for all $t \in (0,1)$.

Since $\phi(O)$ is contained in $\mathbb{R}^+$, it is clear that $\Delta \geq 0$ on $[0,1]$, and thus Ψ'' is non negative on [0,1]. Applying Theorem 2.2 to the function Ψ, we obtain

$$\left|\Upsilon(z) - \frac{1}{d-c}\int_c^d \Psi(u)du\right| \leq \frac{\max\{|\Psi''(c)|, |\Psi''(z)|\}(z-c)^3 + \max\{|\Psi''(d)|, |\Psi''(z)|\}(d-z)^3}{3(d-c)},$$

where

$$\Upsilon(z) = \frac{\Psi'(z)\left((z-c)^2 - (d-z)^2\right) + 2\Psi(c)(z-c) + 2\Psi(d)(d-z)}{2(d-c)}$$

for all $z \in [c,d]$.

Writing these relations in terms of function ϕ, we get the required conclusion. $\square$

Remark. *By putting $z = \frac{c+d}{2}$ in (4.2), we get*

$$\Bigg|\frac{1}{2}\int_0^c\int_0^y \phi((1-x)\alpha+x\beta)dxdy+\frac{1}{2}\int_0^d\int_0^y \phi((1-x)\alpha+x\beta)dxdy$$
$$-\frac{1}{d-c}\int_c^d\int_0^u\int_0^y \phi((1-x)\alpha+x\beta)dxdydu\Bigg|$$
$$\leq \frac{(d-c)^2}{24}\Bigg[\max\left\{|\phi((1-c)\alpha+c\beta))|,\left|\phi\left(\left(1-\frac{c+d}{2}\right)\alpha+\frac{c+d}{2}\beta\right)\right|\right\}$$
$$+\max\left\{|\phi((1-d)\alpha+d\beta)|,\left|\phi\left(\left(1-\frac{c+d}{2}\right)\alpha+\frac{c+d}{2}\beta\right)\right|\right\}\Bigg]. \tag{4.2}$$

Remark. *Analogously, we can give extension of other results, but here we omit the details.*

5. Application to means

In this section we present applications of our main results obtained in section 2 to the following means.

(1) The arithmetic mean

$$A = A(c,d) = \frac{c+d}{2} \quad (c,d>0);$$

(2) The harmonic mean

$$H = H(c,d) = \frac{2}{\frac{1}{c}+\frac{1}{d}} \quad (c,d>0);$$

(3) The logarithmic mean

$$L(c,d) = \frac{d-c}{\log d - \log c} \quad (c,d>0, c\neq d);$$

(4) The generlized logarithmic mean

$$L_n(c,d) = \left[\frac{d^{n+1}-c^{n+1}}{(d-c)(n+1)}\right]^{\frac{1}{n}} \quad (n\in\mathbb{Z}\setminus\{-1,0\},\quad c,d>0, c\neq d).$$

The following propositions are valid in the light of the above results.

Proposition 5.1. *Let $c,d\in\mathbb{R}^+$ with $c<d$ and n be an integer with $|n(n-1)|\geq 3$, then for $q>1, p^{-1}+q^{-1}=1$, we have*

$$|A(c^n,d^n) - L_n^n(c,d)|$$
$$\leq \frac{n(n-1)(d-c)^2}{24}\left[\max\left\{c^{n-2},\left(\frac{c+d}{2}\right)^{n-2}\right\}+\max\left\{\left(\frac{c+d}{2}\right)^{n-2},d^{n-2}\right\}\right], \tag{5.1}$$

$$|A(c^n,d^n)-L_n^n(c,d)| \le M^{\frac{1}{p}}\ \frac{n(n-1)(d-c)^2}{16}\left[\max\left\{c^{(n-2)q},\left(\frac{c+d}{2}\right)^{(n-2)q}\right\}+\max\left\{\left(\frac{c+d}{2}\right)^{(n-2)q},d^{n-2}\right\}\right], \tag{5.2}$$

where $M=\Gamma(1/2)\Gamma(p+1)/[2\Gamma(p+3/2)]$.

Proof. Let $\phi(z)=z^n$ and $z>0$. Then the desired results follow directly from Remarks 2 and 2. □

Proposition 5.2. *Let* $c,d\in\mathbb{R}^+$ *with* $c<d$ *and* n *be an integer with* $|n(n-1)|\ge 3$, *then for* $q>1, p^{-1}+q^{-1}=1$, *we have*

$$|H^{-1}(c^n,d^n)-L_n^n(c^{-1},d^{-1})| \le \frac{n(n-1)(d^{-1}-c^{-1})^2}{24} \times\left[\max\left\{c^{2-n},\left(\frac{c^{-1}+d^{-1}}{2}\right)^{n-2}\right\}+\max\left\{\left(\frac{c^{-1}+d^{-1}}{2}\right)^{n-2},d^{2-n}\right\}\right],$$

$$|H^{-1}(c^n,d^n)-L_n^n(c^{-1},d^{-1})| \le M^{\frac{1}{p}}\ \frac{n(n-1)(d-c)^2}{16} \times\left[\max\left\{c^{(2-n)q},\left(\frac{c^{-1}+d^{-1}}{2}\right)^{(n-2)q}\right\}+\max\left\{\left(\frac{c^{-1}+d^{-1}}{2}\right)^{(n-2)q},d^{2-n}\right\}\right],$$

where $M=\Gamma(1/2)\Gamma(p+1)/[2\Gamma(p+3/2)]$.

Proof. The proof is completed by substituting $a\to a^{-1}$ and $b\to b^{-1}$ in the inequalities (5.1) and (5.2). □

Proposition 5.3. *Let* $c,d\in\mathbb{R}^+$ *with* $c<d$ *and* n *be an integer with* $|n(n-1)|\ge 3$, *then for* $q\ge 1$, *we have*

$$|A(c^n,d^n)-L_n^n(c,d)| \le \frac{n(n-1)(d-c)^2}{24}\left[\left(\max\left\{c^{(n-2)q},\left(\frac{c+d}{2}\right)^{(n-2)q}\right\}\right)^{\frac{1}{q}} +\left(\max\left\{\left(\frac{c+d}{2}\right)^{(n-2)q},d^{(n-2)q}\right\}\right)^{\frac{1}{q}}\right]. \tag{5.3}$$

Proof. Let $\phi(z)=z^n$ and $z>0$. Then the desired results follow directly from Remark 2. □

Proposition 5.4. *Let* $c,d\in\mathbb{R}^+$ *with* $c<d$, $q>1$ *and* $p^{-1}+q^{-1}=1$, *then we have*

$$|A(c^{-1},d^{-1})-L^{-1}(c,d)| \le \frac{(d-c)^2}{12}\left[\max\left\{c^{-3},\ 8(c+d)^{-3}\right\}+\max\left\{d^{-3},\ 8(c+d)^{-3}\right\}\right],$$

$$|A(c^{-1},d^{-1})-L^{-1}(c,d)| \\ \leq M^{\frac{1}{p}}\frac{(d-c)^2}{8}\left[\left(\max\left\{c^{-3q},8^{-q}(c+d)^{-3q}\right\}\right)^{\frac{1}{q}}+\left(\max\left\{d^{-3q},8^{-q}(c+d)^{-3q}\right\}\right)^{\frac{1}{q}}\right],$$

where $M=\Gamma(1/2)\Gamma(p+1)/[2\Gamma(p+3/2)]$.

Proof. Let $\phi(z)=\frac{1}{z}$, $z>0$. Then the desired results follow directly from Remarks 2 and 2. □

Proposition 5.5. *Let $c,d\in\mathbb{R}^+$ with $c<d$, then for all $q\geq 1$ we have*

$$|A(c^{-1},d^{-1})-L^{-1}(c,d)|\leq\frac{(d-c)^2}{12}\left[\left(\max\left\{|c^{-3q},8^{-q}(c+d)^{-3q}\right\}\right)^{\frac{1}{q}}\right. \\ \left.+\left(\max\left\{|d^{-3q},8^{-q}(c+d)^{-3q}\right\}\right)^{\frac{1}{q}}\right].$$

Proof. Let $\phi(z)=\frac{1}{z}$, $z>0$. Then the desired results follow directly from Remark 2. □

Proposition 5.6. *Let $c,d\in\mathbb{R}^+$ with $c<d$ and n be an integer with $|n(n-1)|\geq 3$, then for $q\geq 1$, we have*

$$|H^{-n}(c,d)-L_n^n(c^{-1},d^{-1})| \\ \leq\frac{n(n-1)(d^{-1}-c^{-1})^2}{24}\left[\left(\max\left\{c^{(2-n)q},\left(\frac{c^{-1}+d^{-1}}{2}\right)^{(n-2)q}\right\}\right)^{\frac{1}{q}}\right. \\ \left.+\left(\max\left\{\left(\frac{c^{-1}+d^{-1}}{2}\right)^{(n-2)q},d^{(2-n)q}\right\}\right)^{\frac{1}{q}}\right].$$

Proof. The proof is obvious by substituting $a\to a^{-1}$ and $b\to b^{-1}$ in the inequality (5.3). □

6. Some Error Estimates for Trapezoidal Formula

Let $p=\{z_1,z_2,..,z_n\}$, $z_i\in[c,d]$, $i=\overline{1,n}$ with $c=z_0$, $z_n=d$ and $z_i<z_{i+1}$ for $i=\overline{1,n}$.
Then the well known quadrature formula for the partition p is

$$\int_c^d\phi(z)dz=\tau(\phi,p)+E(\phi,p),$$

where

$$\tau(\phi,p)=\sum_{i=0}^{n-1}\frac{\phi(z_i)+\phi(z_{i+1})}{2}(z_{i+1}-z_i)$$

denotes the trapezoidal formula and $E(\phi,p)$ represents the error approximation associated to it.

Proposition 6.1. *Let the all the requisites of Lemma 2.1 hold. Additionally, if $|\phi''|$ is quasi convex on $[c,d]$, then for all $z_i \in [c,d]$, we have*

$$|E(\phi,p)| \leq \sum_{i=0}^{n-1} \frac{(z_{i+1}-z_i)^3}{24}\left[\max\left\{|\phi''(z_i)|, \left|\phi''\left(\frac{z_i+z_{i+1}}{2}\right)\right|\right\}\right.$$
$$\left. + \max\left\{|\phi''(z_{i+1})|, \left|\phi''\left(\frac{z_i+z_{i+1}}{2}\right)\right|\right\}\right].$$

Proof. To prove the required result, we consider the subintervals $[z_i, z_{i+1}]$ $(i = \overline{0, n-1})$ of the partition p and then applying Remark 2, gives

$$\left|\frac{\phi(z_i)+\phi(z_{i+1})}{2} - \frac{\int_{z_i}^{z_{i+1}} \phi(z)dz}{z_{i+1}-z_i}\right| \leq \frac{(z_{i+1}-z_i)^2}{24}\left[\max\left\{|\phi''(z_i)|, \left|\phi''\left(\frac{z_i+z_{i+1}}{2}\right)\right|\right\}\right.$$
$$\left. + \max\left\{|\phi''(z_{i+1})|, \left|\phi''\left(\frac{z_i+z_{i+1}}{2}\right)\right|\right\}\right],$$

hence from above

$$\begin{aligned}
\left|\int_c^d \phi(z)dz - \tau(\phi,p)\right| &= \left|\sum_{i=0}^{n-1}\left\{\int_{z_i}^{z_{i+1}} \phi(z)dz - \frac{\phi(z_i)+\phi(z_{i+1})}{2}(z_{i+1}-z_i)\right\}\right| \\
&\leq \sum_{i=0}^{n-1}\left|\left\{\int_{z_i}^{z_{i+1}} \phi(z)dz - \frac{\phi(z_i)+\phi(z_{i+1})}{2}(z_{i+1}-z_i)\right\}\right| \\
&\leq \sum_{i=0}^{n-1} \frac{(z_{i+1}-z_i)^3}{24}\left[\max\left\{|\phi''(z_i)|, \left|\phi''\left(\frac{z_i+z_{i+1}}{2}\right)\right|\right\}\right. \\
&\quad \left. + \max\left\{|\phi''(z_{i+1})|, \left|\phi''\left(\frac{z_i+z_{i+1}}{2}\right)\right|\right\}\right].
\end{aligned}$$

□

Proposition 6.2. *Let the all the requisites of Lemma 2.1 hold. Additionally, if $|\phi''|^q$ is quasi convex for $q > 1$ and $p^{-1} + q^{-1} = 1$, then we have*

$$|E(\phi,p)| \leq M^{\frac{1}{p}} \sum_{i=0}^{n-1} \frac{(z_{i+1}-z_i)^3}{16}\left[\left(\max\left\{|\phi''(z_i)|^q, \left|\phi''\left(\frac{z_i+z_{i+1}}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\max\left\{|\phi''(z_{i+1})|^q, \left|\phi''\left(\frac{z_i+z_{i+1}}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right], \tag{6.1}$$

where $M = \Gamma(1/2)\Gamma(p+1)/[2\Gamma(p+3/2)]$.

Proof. Proceeding on the same lines like in the Proposition 6.1, we can prove the inequality in (6.1) by making use of Remark 2. □

Proposition 6.3. *Let the all the requisites of Lemma 2.1 hold. Additionally, if $|\phi''|^q$ is quasi convex for $q \geq 1$, then we have*

$$|E(\phi,p)| \leq \sum_{i=0}^{n-1} \frac{(z_{i+1}-z_i)^3}{24} \left[\left(\max\left\{ |\phi''(z_i)|^q, \left|\phi''\left(\frac{z_i+z_{i+1}}{2}\right)\right|^q \right\} \right)^{\frac{1}{q}} + \left(\max\left\{ |\phi''(z_{i+1})|^q, \left|\phi''\left(\frac{z_i+z_{i+1}}{2}\right)\right|^q \right\} \right)^{\frac{1}{q}} \right]. \tag{6.2}$$

Proof. Proceeding on the same lines like in the Proposition 6.1, we can prove the inequality in (6.2) by making use of Remark 2. □

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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