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WEIGHTED OSTROWSKI AND GRÜSS TYPE INEQUALITIES

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Abstract. The main objective of this paper is to present several weighted Ostrowski and Grüss type inequalities for continuous functions with one point of nondifferentiability using certain inequalities for the Chebyshev functional.

1. Introduction

For two real functions $f, g : [a, b] \to \mathbf{R}$ such that $f, g, f \cdot g \in L_1[a, b]$, the Chebyshev functional [12] is defined by

$$S(f,g) = \frac{1}{b-a}\int_a^b f(t)\,g(t)\,\mathrm{d}t - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\,\frac{1}{b-a}\int_a^b g(t)\,\mathrm{d}t.$$

The symbol $L_p[a, b]$, $1 \le p < \infty$, denotes the space of p-power integrable functions on interval $[a, b]$ equipped with the norm $\|f\|_p = \left(\int_a^b |f(t)|^p\,\mathrm{d}t\right)^{1/p}$ and $L_\infty[a, b]$ stands for the space of all essentially bounded functions on interval $[a, b]$ with the norm $\|f\|_\infty = \text{ess sup}_{t\in[a,b]} |f(t)|$.

Under the assumptions that $f, g : [a, b] \to \mathbb{R}$ are two bounded integrable functions, α, β, γ and δ are real numbers such that $\alpha \le f(t) \le \beta$, and $\gamma \le g(t) \le \delta$, for all $t \in [a, b]$, Grüss proved the following inequality which establishes a connection between the integral of the product and the product of the integrals (see [12], p.296.)

$$|S(f,g)| \le \frac{1}{4}(\beta-\alpha)(\delta-\gamma).$$

Another representation for $S(f, g)$ was obtained by Sonin (see [12], p.246)

$$S(f,g) = \frac{1}{b-a}\int_a^b (g(t)-\eta)\left(f(t) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right)\mathrm{d}t, \tag{1.1}$$

where η is an arbitrary real number.

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In 1938 A.M.Ostrowski [14] pointed out the following inequality which gives an approximation of the integral $\frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s$ as follows

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s \right| \leq \frac{(x-a)^2+(b-x)^2}{2(b-a)}C, \tag{1.2}$$

for all $x \in [a,b]$, where $f : [a,b] \to \mathbf{R}$ is continuous on $[a,b]$ and differentiable on (a,b) such that $|f'(x)| \leq C$.

Over the last decades some new Ostrowski type inequalities and Grüss type inequalities have been studied and applied in numerical analysis (see [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15] and the references cited therein).

In paper [13] Niezgoda proved Ostrowski type inequalities for functions whose the first derivatives are from the $\mathcal{D}(x_0)$ class. The symbol $\mathcal{D}(x_0)$, $x_0 \in [a,b]$, denotes the class of all continuous functions $f : [a,b] \to \mathbb{R}$ differentiable on the set $(a,x_0) \cup (x_0,b)$ such that

$$\sup_{t\in(a,x_0)} |f'(t)| < \infty \text{ and } \sup_{t\in(x_0,b)} |f'(t)| < \infty. \tag{1.3}$$

In what follows, (p,q) is a pair of conjugate exponents if $1 \leq p,q \leq \infty$ and $\frac{1}{p}+\frac{1}{q}=1$, with the convention $1/\infty = 0$ and $1/0 = \infty$. Further, $I \subset \mathbf{R}$ is an interval, Int$I$ is the interior of I and $J(f)$ denote the mean value of f

$$J(f) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t. \tag{1.4}$$

Theorem 1.1. *Let $f : I \to \mathbf{R}$ be a differentiable mapping on IntI and let $[a,b] \subset$ IntI. Suppose that $f' \in \mathcal{D}(x_0)$, for some $x_0 \in [a,b]$,*

$$M_l = \sup_{t\in(a,x_0)} |f''(t)| < \infty \text{ and } M_r = \sup_{t\in(x_0,b)} |f''(t)| < \infty, \tag{1.5}$$

where $M_l = 0$ if $x_0 = a$ and $M_r = 0$ if $x_0 = b$. Then for $x \in [a,b]$ the following three inequalities hold

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a} \right|$$
$$\leq \begin{cases} \frac{b-a}{2(p+1)^{1/p}(q+1)^{1/q}}\left\{[M_l(x_0-a)]^p\frac{x_0-a}{b-a} + [M_r(b-x_0)]^p\frac{b-x_0}{b-a}\right\}^{1/p}, & 1<p<\infty, \\ \frac{1}{4}\left[M_l(x_0-a)^2 + M_r(b-x_0)^2\right], & p=1, \\ \frac{1}{4}(b-a)\max\{M_l(x_0-a), M_r(b-x_0)\}, & p=\infty, \end{cases} \tag{1.6}$$

where (p,q) is a pair of conjugate exponents.

Using a method originated in [10], Niezgoda [13] established the following Grüss type inequalities.

Theorem 1.2. *Suppose that $f \in \mathcal{D}(x_0)$, for some $x_0 \in [a,b]$, (p,q) is a pair of conjugate exponents and $g \in L_q[a,b]$. Denote*

$$M_l = \sup_{t\in(a,x_0)} |f'(t)| < \infty \text{ and } M_r = \sup_{t\in(x_0,b)} |f'(t)| < \infty, \tag{1.7}$$

where $M_l = 0$ *for* $x_0 = a$ *and* $M_r = 0$ *for* $x_0 = b$. *Then the following inequalities hold*

$$|S(f,g)| \leq \begin{cases} \frac{1}{(b-a)(p+1)^{1/p}} \left[M_l^p (x_0 - a)^{p+1} + M_r^p (b - x_0)^{p+1} \right]^{1/p} \|g - J(g)\|_q, & 1 \leq p < \infty, \\ \frac{1}{(b-a)} \max\{M_l (x_0 - a), M_r (b - x_0)\} \|g - J(g)\|_1, & p = \infty. \end{cases} \tag{1.8}$$

Aglić et al. have proved the following identity (see [2]) by using generalizations of the Montgomery identity [1]:

Theorem 1.3. *Assume that* I *is an open interval in* $\mathbf{R}$, $[a,b] \subset I$, *and* $w : [a,b] \to [0,\infty)$ *is some probability density function, that is integrable function satisfying* $\int_a^b w(t)\,\mathrm{d}t = 1$, *and* $W(t) = \int_a^t w(x)\,\mathrm{d}x$ *for* $t \in [a,b]$, $W(t) = 0$ *for* $t < a$ *and* $W(t) = 1$ *for* $t > b$. *Suppose* $f : I \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is absolutely continuous for some* $n \geq 1$. *Then for each* $x \in \left[a, \frac{a+b}{2}\right]$ *the following identity holds*

$$\begin{aligned} \int_a^b w(t)f(t)\mathrm{d}t = \frac{1}{2} \Bigg[& \sum_{i=0}^{n-1} \frac{f^{(i)}(x)}{i!} \int_a^b w(s)(s-x)^i \mathrm{d}s \\ & + \sum_{i=0}^{n-1} \frac{f^{(i)}(a+b-x)}{i!} \int_a^b w(s)(s-a-b+x)^i \mathrm{d}s \Bigg] \\ & + \frac{1}{(n-1)!} \int_a^b F_{w,n}(x,s) f^{(n)}(s)\mathrm{d}s, \end{aligned} \tag{1.9}$$

where

$$F_{w,n}(x,s) = \begin{cases} -\int_a^s w(u)(u-s)^{n-1}\,\mathrm{d}u, & a \leq s \leq x, \\ -\frac{1}{2}\left[\int_a^s w(u)(u-s)^{n-1}\,du - \int_s^b w(u)(u-s)^{n-1}\,\mathrm{d}u\right], & x < s \leq a+b-x, \\ \int_s^b w(u)(u-s)^{n-1}\,\mathrm{d}u, & a+b-x < s \leq b. \end{cases}$$

In this paper, using results from paper [2], we will obtain the weighted generalization of inequalities (1.6) involving derivatives of arbitrary order of function f. As special cases, we get some new error estimates for two-point quadrature formulae.

2. Main results

We introduce the following notation

$$\begin{aligned} D_w(f;x) = \int_a^b w(t)f(t)\mathrm{d}t - \frac{1}{2} \Bigg[& \sum_{i=0}^{n-1} \frac{f^{(i)}(x)}{i!} \int_a^b w(s)(s-x)^i \mathrm{d}s \\ & + \sum_{i=0}^{n-1} \frac{f^{(i)}(a+b-x)}{i!} \int_a^b w(s)(s-a-b+x)^i \mathrm{d}s \Bigg] \end{aligned} \tag{2.1}$$

and

$$D(f;x) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{1}{2}\sum_{i=0}^{n-1}\left[f^{(i)}(x) + (-1)^i f^{(i)}(a+b-x)\right]\frac{(b-x)^{i+1}-(a-x)^{i+1}}{(i+1)!\,(b-a)}. \quad (2.2)$$

In the next theorem, under the assumptions that $f^{(n-1)} \in \mathcal{D}(x_0)$, we will consider the upper bound for $D_w(f;x)$.

Theorem 2.1. *Assume that $f : I \to \mathbf{R}$ is a differentiable mapping on IntI, $[a,b] \subset$ IntI and $w : [a,b] \to [0,\infty)$ is some probability density function. Suppose that $f^{(n-1)} \in \mathcal{D}(x_0)$, for some $x_0 \in [a,b]$, $n \in \mathbf{N}$ and*

$$C_l = \sup_{t\in(a,x_0)} |f^{(n)}(t)| < \infty \text{ and } C_r = \sup_{t\in(x_0,b)} |f^{(n)}(t)| < \infty,$$

where $C_l = 0$ if $x_0 = a$ and $C_r = 0$ if $x_0 = b$. Then for $x \in \left[a, \frac{a+b}{2}\right]$ the following inequalities hold

$$|D_w(f;x)| \le \begin{cases} \frac{1}{(n-1)!}\left[C_l^p(x_0-a) + C_r^p(b-x_0)\right]^{1/p}\|F_{w,n}(x,\cdot)\|_q, & 1 \le p < \infty, \\ \frac{1}{(n-1)!}\max\{C_l, C_r\}\|F_{w,n}(x,\cdot)\|_1, & p = \infty, \end{cases} \quad (2.3)$$

where (p,q) is a pair of conjugate exponents.

Proof. Using identity (1.9) and notation (2.1) we get

$$D_w(f;x) = \frac{1}{(n-1)!}\int_a^b F_{w,n}(x,s)f^{(n)}(s)\mathrm{d}s = \frac{1}{(n-1)!}\left[\int_a^{x_0} F_{w,n}(x,s)f^{(n)}(s)\mathrm{d}s + \int_{x_0}^b F_{w,n}(x,s)f^{(n)}(s)\mathrm{d}s\right].$$

Now, taking the absolute value and applying triangle inequality and Hölder inequality, respectively, we have

$$\begin{aligned} |D_w(f;x)| &\le \frac{1}{(n-1)!}\left[\left|\int_a^{x_0} F_{w,n}(x,s)f^{(n)}(s)\mathrm{d}s\right| + \left|\int_{x_0}^b F_{w,n}(x,s)f^{(n)}(s)\mathrm{d}s\right|\right] \\ &\le \frac{1}{(n-1)!}\left[\|F_{w,n}(x,\cdot)\|_{q,[a,x_0]}\left\|f^{(n)}\right\|_{p,[a,x_0]} + \|F_{w,n}(x,\cdot)\|_{q,[x_0,b]}\left\|f^{(n)}\right\|_{p,[x_0,b]}\right] \\ &\le \frac{1}{(n-1)!}\left[C_l(x_0-a)^{1/p}\|F_{w,n}(x,\cdot)\|_{q,[a,x_0]} + C_r(b-x_0)^{1/p}\|F_{w,n}(x,\cdot)\|_{q,[x_0,b]}\right]. \end{aligned}$$

Further, using discrete Hölder inequality for $1 \le p < \infty$ we obtain the following inequality

$$|D_w(f;x)| \le \frac{1}{(n-1)!}\left[C_l^p(x_0-a) + C_r^p(b-x_0)\right]^{1/p}\|F_{w,n}(x,\cdot)\|_q.$$

For $p = \infty$ we get

$$|D_w(f;x)| \le \max\{C_l, C_r\}\|F_{w,n}(x,\cdot)\|_1,$$

which completes the proof. □

In the next corollary we will use the Beta function and the incomplete Beta function of Euler type defined by

$$B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}\,\mathrm{d}t, \qquad B_r(u,v)=\int_0^r t^{u-1}(1-t)^{v-1}\,\mathrm{d}t, \qquad u,v>0.$$

Corollary 2.2. *With the assumptions of Theorem 2.1 for each* $x\in\left[a,\frac{a+b}{2}\right]$ *we have*

$$\begin{aligned}&|D(f;x)|\\ &\le \frac{1}{n!(b-a)}\max\left\{(x-a)^n,\frac{(a-x)^n+(b-x)^n}{2}\right\}\left[C_l(x_0-a)+C_r(b-x_0)\right],\end{aligned} \tag{2.4}$$

$$\begin{aligned}&|D(f;x)|\\ &\le \frac{1}{n!}\left(\frac{3(x-a)^{2n+1}+(b-x)^{2n+1}}{2(2n+1)(b-a)^2}\right.\\ &\left.+\frac{(-1)^n(b-a)^{2n-1}}{2}\left[B_{\frac{b-x}{b-a}}(n+1,n+1)-B_{\frac{x-a}{b-a}}(n+1,n+1)\right]\right)^{1/2}\\ &\times\left[C_l^2(x_0-a)+C_r^2(b-x_0)\right]^{1/2},\end{aligned} \tag{2.5}$$

$$\begin{aligned}&|D(f;x)|\\ &\le \frac{1}{(n+1)!}\left(\frac{(x-a)^{n+1}\left[2+(-1)^{n+1}\right]+(b-x)^{n+1}}{(b-a)}-\left(\frac{b-a}{2}\right)^n\left[\frac{(-1)^{n+1}+1}{2}\right]\right)\\ &\times\max\{C_l,\,C_r\}.\end{aligned} \tag{2.6}$$

Proof. Applying (2.3) with $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ and $p=1,p=2,p=\infty$, respectively, we get above inequalities. □

Remark. *For* $n=1$ *inequality (2.4) reduces to the following inequality for two-point quadrature formula*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(x)+f(a+b-x)}{2}\right|\\ &\le\left(\frac{1}{4}+\frac{|3a+b-4x|}{4(b-a)}\right)\left[C_l(x_0-a)+C_r(b-x_0)\right].\end{aligned}$$

Since, for $n=2$ *we obtain the following inequality*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(x)+f(a+b-x)}{2}-\frac{f'(x)-f'(a+b-x)}{2}\left(\frac{a+b}{2}-x\right)\right|\\ &\le\left(\frac{(x-a)^2+(b-x)^2}{4(b-a)}\right)\left[C_l(x_0-a)+C_r(b-x_0)\right].\end{aligned}$$

Remark. *For $n=1$ inequality (2.6) reduces to the following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(x)+f(a+b-x)}{2}\right|$$
$$\leq\left(\frac{3(x-a)^2+(b-x)^2}{2(b-a)}-\frac{b-a}{4}\right)\max\{C_l,\,C_r\}.$$

Further, for $n=2$ from inequality (2.6) we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(x)+f(a+b-x)}{2}-\frac{f'(x)-f'(a+b-x)}{2}\left(\frac{a+b}{2}-x\right)\right|$$
$$\leq\left(\frac{(x-a)^3+(b-x)^3}{6(b-a)}\right)\max\{C_l,\,C_r\}.$$

Theorem 2.3. *Suppose that $f:I\to\mathbf{R}$ is a differentiable mapping on IntI, $[a,b]\subset$ IntI and $w:[a,b]\to[0,\infty)$ is some probability density function. Assume that $f^{(n)}\in\mathcal{D}(x_0)$, for some $n\in\mathbf{N}$ and*

$$C_l=\sup_{t\in(a,x_0)}|f^{(n+1)}(t)|<\infty\ \ \text{and}\ \ C_r=\sup_{t\in(x_0,b)}|f^{(n+1)}(t)|<\infty,$$

where $C_l=0$ for $x_0=a$ and $C_r=0$ for $x_0=b$. Then for $x\in\left[a,\frac{a+b}{2}\right]$ the following inequalities hold

$$\left|D_w(f;x)-\frac{1}{(n-1)!}f^{(n)}(x_0)\int_a^b F_{w,n}(x,s)\mathrm{d}s\right|$$
$$\leq\begin{cases}\frac{1}{(n-1)!}\left[\frac{C_l^p(x_0-a)^{p+1}}{p+1}+\frac{C_r^p(b-x_0)^{p+1}}{p+1}\right]^{1/p}\|F_{w,n}(x,\cdot)\|_q, & 1\leq p<\infty,\\ \frac{1}{(n-1)!}\max\{C_l(x_0-a),\,C_r(b-x_0)\}\|F_{w,n}(x,\cdot)\|_1, & p=\infty,\end{cases}\tag{2.7}$$

where (p,q) is a pair of conjugate exponents.

Proof. Using identity (1.9) and notation (2.1) we get

$$D_w(f;x)-\frac{1}{(n-1)!}f^{(n)}(x_0)\int_a^b F_{w,n}(x,s)\mathrm{d}s$$
$$=\frac{1}{(n-1)!}\int_a^b F_{w,n}(x,s)\left(f^{(n)}(s)-f^{(n)}(x_0)\right)\mathrm{d}s.$$

Further, taking the absolute value, applying the triangle inequality and Hölder inequality we obtain

$$\begin{aligned}&\left|D_w(f;x)-\frac{1}{(n-1)!}f^{(n)}(x_0)\int_a^b F_{w,n}(x,s)\mathrm{d}s\right|\\&\le\frac{1}{(n-1)!}\left[\left|\int_a^{x_0}F_{w,n}(x,s)\left(f^{(n)}(s)-f^{(n)}(x_0)\right)\mathrm{d}s\right|\right.\\&\left.+\left|\int_{x_0}^b F_{w,n}(x,s)\left(f^{(n)}(s)-f^{(n)}(x_0)\right)\mathrm{d}s\right|\right]\\&\le\frac{1}{(n-1)!}\left[\|F_{w,n}(x,\cdot)\|_{q,[a,x_0]}\left\|f^{(n)}(\cdot)-f^{(n)}(x_0)\right\|_{p,[a,x_0]}\right.\\&\left.+\|F_{w,n}(x,\cdot)\|_{q,[x_0,b]}\left\|f^{(n)}(\cdot)-f^{(n)}(x_0)\right\|_{p,[x_0,b]}\right].\end{aligned}$$

By Lagrange mean value theorem there exist $\mu\in\langle s,x_0\rangle$, such that

$$f^{(n)}(s)-f^{(n)}(x_0)=f^{(n+1)}(\mu)(s-x_0), \tag{2.8}$$

from which follows

$$\left|f^{(n)}(s)-f^{(n)}(x_0)\right|\le C_l(x_0-s),$$

thus, for $1\le p<\infty$, we get

$$\left\|f^{(n)}(\cdot)-f^{(n)}(x_0)\right\|_{p,[a,x_0]}^p\le\frac{C_l^p(x_0-a)^{p+1}}{p+1}. \tag{2.9}$$

Similarly, we deduce

$$\left\|f^{(n)}(\cdot)-f^{(n)}(x_0)\right\|_{p,[x_0,b]}^p\le\frac{C_r^p(b-x_0)^{p+1}}{p+1}. \tag{2.10}$$

Further, for $p=\infty$ the following inequalities hold:

$$\begin{aligned}\left\|f^{(n)}(\cdot)-f^{(n)}(x_0)\right\|_{\infty,[a,x_0]}&\le C_l(x_0-a),\\\left\|f^{(n)}(\cdot)-f^{(n)}(x_0)\right\|_{\infty,[x_0,b]}&\le C_r(b-x_0).\end{aligned} \tag{2.11}$$

Finally, using discrete Hölder inequality and inequalities (2.9) and (2.10) for $1\le p<\infty$ we get

$$\begin{aligned}&\left|D_w(f;x)-\frac{1}{(n-1)!}f^{(n)}(x_0)\int_a^b F_{w,n}(x,s)\mathrm{d}s\right|\\&\le\frac{1}{(n-1)!}\left[\left(\left\|f^{(n)}(\cdot)-f^{(n)}(x_0)\right\|_{p,[a,x_0]}^p+\left\|f^{(n)}(\cdot)-f^{(n)}(x_0)\right\|_{p,[x_0,b]}^p\right)^{1/p}\right.\\&\left.\times\left(\|F_{w,n}(x,\cdot)\|_{q,[a,x_0]}^q+\|F_{w,n}(x,\cdot)\|_{q,[x_0,b]}^q\right)^{1/q}\right]\\&\le\frac{1}{(n-1)!}\left[\frac{C_l^p(x_0-a)^{p+1}}{p+1}+\frac{C_r^p(b-x_0)^{p+1}}{p+1}\right]^{1/p}\|F_{w,n}(x,\cdot)\|_q.\end{aligned}$$

Similar, for $p=\infty$ we obtain

$$\begin{aligned}&\left|D_w\left(f;x\right)-\frac{1}{(n-1)!}f^{(n)}\left(x_0\right)\int_a^b F_{w,n}(x,s)\mathrm{d}s\right|\\&\leq\max\left\{C_l\left(x_0-a\right),C_r\left(b-x_0\right)\right\}\left\|F_{w,n}\left(x,\cdot\right)\right\|_1,\end{aligned}$$

which completes the proof. □

Corollary 2.4. *With the assumptions of Theorem 2.3 for each* $x\in\left[a,\frac{a+b}{2}\right]$ *we have*

$$\begin{aligned}&\left|D\left(f;x\right)-\frac{(-1)^n+1}{2}\,\frac{(x-a)^{n+1}+(-1)^n\left(b-x\right)^{n+1}}{(n+1)!\left(b-a\right)}f^{(n)}\left(x_0\right)\right|\\&\leq\frac{1}{2n!\left(b-a\right)}\max\left\{(x-a)^n,\frac{(a-x)^n+(b-x)^n}{2}\right\}\left[C_l\left(x_0-a\right)^2+C_r\left(b-x_0\right)^2\right],\end{aligned}\tag{2.12}$$

$$\begin{aligned}&\left|D\left(f;x\right)-\frac{(-1)^n+1}{2}\,\frac{(x-a)^{n+1}+(-1)^n\left(b-x\right)^{n+1}}{(n+1)!\left(b-a\right)}f^{(n)}\left(x_0\right)\right|\\&\leq\frac{1}{3n!}\left(\frac{3\left(x-a\right)^{2n+1}+\left(b-x\right)^{2n+1}}{2\left(2n+1\right)\left(b-a\right)^2}\right.\\&\left.+\frac{(-1)^n\left(b-a\right)^{2n-1}}{2}\left[B_{\frac{b-x}{b-a}}\left(n+1,n+1\right)-B_{\frac{x-a}{b-a}}\left(n+1,n+1\right)\right]\right)^{1/2}\\&\times\left[C_l^2\left(x_0-a\right)^3+C_r^2\left(b-x_0\right)^3\right]^{1/2},\end{aligned}\tag{2.13}$$

$$\begin{aligned}&\left|D\left(f;x\right)-\frac{(-1)^n+1}{2}\,\frac{(x-a)^{n+1}+(-1)^n\left(b-x\right)^{n+1}}{(n+1)!\left(b-a\right)}f^{(n)}\left(x_0\right)\right|\\&\leq\frac{1}{(n+1)!}\left(\frac{(x-a)^{n+1}\left[2+(-1)^{n+1}\right]+(b-x)^{n+1}}{(b-a)}-\left(\frac{b-a}{2}\right)^n\left[\frac{(-1)^{n+1}+1}{2}\right]\right)\\&\times\max\{C_l\left(x_0-a\right),\,C_r\left(b-x_0\right)\}.\end{aligned}\tag{2.14}$$

Proof. Applying (2.7) with $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ and $p=1,p=2,p=\infty$, respectively, we get above inequalities. □

Remark. *For $n=1$ inequalities (2.12) and (2.14) reduce to the following inequalities for two-point quadrature formula*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f\left(x\right)+f\left(a+b-x\right)}{2}\right|\\&\leq\left(\frac{1}{8}+\frac{|3a+b-4x|}{8\left(b-a\right)}\right)\left[C_l\left(x_0-a\right)^2+C_r\left(b-x_0\right)^2\right]\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(x)+f(a+b-x)}{2}\right|\\&\leq\left(\frac{3(x-a)^2+(b-x)^2}{2(b-a)}-\frac{b-a}{4}\right)\max\{C_l(x_0-a),\,C_r(b-x_0)\}.\end{aligned}$$

Remark. *For twice differentiable function f, $p=q=2$, $x_0=\frac{a+b}{2}$ and $x=a$ inequality (2.13) yields the following trapezoid inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{12\sqrt{3}}(b-a)^2\left[\frac{1}{2}C_l^2+\frac{1}{2}C_r^2\right]^{1/2},\tag{2.15}$$

where $C_l=\sup_{t\in\left(a,\frac{a+b}{2}\right)}|f''(t)|<\infty$ and $C_r=\sup_{t\in\left(\frac{a+b}{2},b\right)}|f''(t)|<\infty$.

Niezgoda [13] *has improved inequality (2.15) with the constant* $1/12$.

At the end, we will establish certain Grüss type inequalities involving derivatives of arbitrary order of function f.

Theorem 2.5. *Let $f:I\to\mathbf{R}$ be a differentiable mapping on IntI, $[a,b]\subset$ IntI and let $w:[a,b]\to[0,\infty)$ be some probability density function. Suppose that $f^{(n)}\in\mathcal{D}(x_0)$, for some $n\in\mathbf{N}$ and*

$$C_l=\sup_{t\in(a,x_0)}|f^{(n+1)}(t)|<\infty\ \text{ and }\ C_r=\sup_{t\in(x_0,b)}|f^{(n+1)}(t)|<\infty.$$

Then for $x\in\left[a,\frac{a+b}{2}\right]$ the following inequalities hold

$$\begin{aligned}&\left|D_w(f;x)-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(n-1)!(b-a)}\int_a^b F_{w,n}(x,s)\mathrm{d}s\right|\\&\leq\begin{cases}\frac{1}{(n-1)!}\left[\frac{C_l^p(x_0-a)^{p+1}}{p+1}+\frac{C_r^p(b-x_0)^{p+1}}{p+1}\right]^{1/p}\left\|F_{w,n}(x,\cdot)-J(F_{w,n})\right\|_q, & 1\leq p<\infty,\\ \frac{1}{(n-1)!}\max\{C_l(x_0-a),\,C_r(b-x_0)\}\left\|F_{w,n}(x,\cdot)-J(F_{w,n})\right\|_1, & p=\infty,\end{cases}\end{aligned}\tag{2.16}$$

where (p,q) is a pair of conjugate exponents.

Proof. By using identity (1.9) and notation (2.1) we obtain

$$\begin{aligned}&D_w(f;x)-\frac{1}{(n-1)!(b-a)}\int_a^b F_{w,n}(x,s)\mathrm{d}s\int_a^b f^{(n)}(s)\,\mathrm{d}s\\&=\frac{1}{(n-1)!}\left(\int_a^b F_{w,n}(x,s)f^{(n)}(s)\mathrm{d}s-\frac{1}{b-a}\int_a^b F_{w,n}(x,s)\mathrm{d}s\int_a^b f^{(n)}(s)\,\mathrm{d}s\right).\end{aligned}$$

Now, applying the Sonin identity (1.1) with $F_{w,n}(x,\cdot)$ instead of f, $f^{(n)}$ instead of g and $\eta = f^{(n)}(x_0)$, we deduce

$$\begin{aligned}
&\int_a^b F_{w,n}(x,s)f^{(n)}(s)\mathrm{d}s - \frac{1}{b-a}\int_a^b F_{w,n}(x,s)\mathrm{d}s \int_a^b f^{(n)}(s)\,\mathrm{d}s \\
&= \int_a^b \left(f^{(n)}(s) - f^{(n)}(x_0)\right)\left(F_{w,n}(x,s) - \frac{1}{b-a}\int_a^b F_{w,n}(x,s)\mathrm{d}s\right)\mathrm{d}s \\
&= \int_a^{x_0} \left(f^{(n)}(s) - f^{(n)}(x_0)\right)\left(F_{w,n}(x,s) - J\left(F_{w,n}\right)\right)\mathrm{d}s \\
&+ \int_{x_0}^b \left(f^{(n)}(s) - f^{(n)}(x_0)\right)\left(F_{w,n}(x,s) - J\left(F_{w,n}\right)\right)\mathrm{d}s.
\end{aligned}$$

Further, we apply the triangle inequality and Hölder inequality to get

$$\begin{aligned}
&\left| D_w\left(f;x\right) - \frac{f^{(n-1)}\left(b\right) - f^{(n-1)}\left(a\right)}{\left(n-1\right)!\left(b-a\right)}\int_a^b F_{w,n}(x,s)\mathrm{d}s \right| \\
&\leq \frac{1}{(n-1)!}\left(\left|\int_a^{x_0} \left(f^{(n)}(s) - f^{(n)}(x_0)\right)\left(F_{w,n}(x,s) - J\left(F_{w,n}\right)\right)\mathrm{d}s\right|\right. \\
&\left. + \left|\int_{x_0}^b \left(f^{(n)}(s) - f^{(n)}(x_0)\right)\left(F_{w,n}(x,s) - J\left(F_{w,n}\right)\right)\mathrm{d}s\right|\right) \\
&\leq \frac{1}{(n-1)!}\left[\left\|F_{w,n}\left(x,\cdot\right) - J\left(F_{w,n}\right)\right\|_{q,[a,x_0]}\left\|f^{(n)}\left(\cdot\right) - f^{(n)}\left(x_0\right)\right\|_{p,[a,x_0]}\right. \\
&\left. + \left\|F_{w,n}\left(x,\cdot\right) - J\left(F_{w,n}\right)\right\|_{q,[x_0,b]}\left\|f^{(n)}\left(\cdot\right) - f^{(n)}\left(x_0\right)\right\|_{p,[x_0,b]}\right].
\end{aligned}$$

For $1 \leq p < \infty$ using discrete Hölder inequality and inequalities (2.9) and (2.10) we deduce

$$\begin{aligned}
&\left| D_w\left(f;x\right) - \frac{f^{(n-1)}\left(b\right) - f^{(n-1)}\left(a\right)}{\left(n-1\right)!\left(b-a\right)}\int_a^b F_{w,n}(x,s)\mathrm{d}s \right| \\
&\leq \frac{1}{(n-1)!}\left[\left\|F_{w,n}\left(x,\cdot\right) - J\left(F_{w,n}\right)\right\|_{q,[a,x_0]}^q + \left\|F_{w,n}\left(x,\cdot\right) - J\left(F_{w,n}\right)\right\|_{q,[x_0,b]}^q\right]^{\frac{1}{q}} \\
&\times\left[\left\|f^{(n)}\left(\cdot\right) - f^{(n)}\left(x_0\right)\right\|_{p,[a,x_0]}^p + \left\|f^{(n)}\left(\cdot\right) - f^{(n)}\left(x_0\right)\right\|_{p,[x_0,b]}^p\right]^{\frac{1}{p}} \\
&\leq \frac{1}{(n-1)!}\left[\frac{C_l^p\left(x_0-a\right)^{p+1}}{p+1} + \frac{C_r^p\left(b-x_0\right)^{p+1}}{p+1}\right]^{1/p}\left\|F_{w,n}\left(x,\cdot\right) - J\left(F_{w,n}\right)\right\|_q.
\end{aligned}$$

Similarlly, for $p = \infty$ we have

$$\begin{aligned}
&\left| D_w\left(f;x\right) - \frac{f^{(n-1)}\left(b\right) - f^{(n-1)}\left(a\right)}{\left(n-1\right)!\left(b-a\right)}\int_a^b F_{w,n}(x,s)\mathrm{d}s \right| \\
&\leq \frac{1}{(n-1)!}\max\{C_l\left(x_0-a\right), C_r\left(b-x_0\right)\}\left\|F_{w,n}\left(x,\cdot\right) - J\left(F_{w,n}\right)\right\|_1.
\end{aligned}$$

□

Corollary 2.6. *Under the assumptions of Theorem 2.5 for each* $x \in \left[a, \frac{a+b}{2}\right]$ *we have*

$$\left| D_w(f;x) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(n-1)!\,(b-a)} \int_a^b F_{w,n}(x,s)\mathrm{d}s \right|$$
$$\leq \begin{cases} \frac{(b-a)^{1+1/p}}{2^{1+1/p}(p+1)^{1/p}(n-1)!} \left[C_l^p + C_r^p\right]^{1/p} \left\| F_{w,n}(x,\cdot) - J(F_{w,n}) \right\|_q, & 1 \leq p < \infty, \\ \frac{b-a}{2(n-1)!} \max\{C_l, C_r\} \left\| F_{w,n}(x,\cdot) - J(F_{w,n}) \right\|_1, & p = \infty, \end{cases} \tag{2.17}$$

Proof. Applying (2.16) with $x_0 = \frac{a+b}{2}$, we obtain inequalities (2.17). □

Remark. *Setting* $w(t) = \frac{1}{b-a}$ *for* $t \in [a,b]$, $n = 1$, $p = q = 2$ *and* $x = a$ *we get trapezoid inequality (2.15).*

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