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A NOTE ON THE ASYMPTOTIC BEHAVIOR OF KUMMER'S HYPERGEOMETRIC FUNCTION WITH LARGE VALUES OF b AND z

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Abstract. This paper provides a proof, based on probabilistic arguments, of an asymptotic property of Kummer's hypergeometric function $M(a, b; z)$, valid when b and z are both large and approach $+\infty$ and $-\infty$, respectively, with the same order. The novelty of the result consists in the fact that, for its validity, no restriction is imposed on (the limiting value of) z/b, except that of being a negative number, whereas all the other known results require significant limitations.

1. Introduction

In 1837, the German mathematician E.E.Kummer introduced the so-called *confluent hypergeometric function of the first type* through the formula

$$M(a, b; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \tag{1.1}$$

where $(x)_n$ is defined by either $(x)_0 := 1$ or $(x)_n := x(x+1)\dots(x+n-1)$ for any $n \in \mathbb{N}$. Other symbols used instead of $M(a, b; z)$ are $\Phi(a, b; z)$ and ${}_1F_1(a, b; z)$. It is well-known that the series converges for all $z \in \mathbb{C}$, provided that $b \neq -m$, with $m \in \mathbb{N}_0$. See, e.g., Chapter 13 of [1] or Chapter VI of [3] for more information. There are also some monographs entirely devoted to hypergeometric functions and their applications, such as [2]. In fact, the relevance of the Kummer function is acknowledged not only in the realm of special functions, but also in various fields of pure and applied mathematics, such as: algebraic geometry, ODE's, mathematical physics, probability theory, stochastic processes, bayesian nonparametrics, and others. For example, it is interesting to note that, in probability theory, Kummer's function is related to the expression of the absolute moments of a Gaussian random variable.

The aim of this paper is to describe the asymptotic behaviour of Kummer's function when b and z are both large and approach $+\infty$ and $-\infty$, respectively, with the same order. Actually, there are many works that have studied the asymptotic

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behavior of $M(a, b; z)$ for large b and z, which typically provide Poincaré-type asymptotic expansions. However, at the best of the author's knowledge, it seems that all these works either exclude that b and z could approach $+\infty$ and $-\infty$, respectively, or they impose further restrictions on $|z/b|$. For example, in Section VI.13.3 of [3], an asymptotic expansion of the above type is given under the constrain that $|z/b| \leq 1 - \varepsilon$, for some $\varepsilon \in (0, 1)$. In Section 13.5.17 of [1], one can find other asymptotic expansions which are valid if $z > 2b - a > 1$, for real a, b and z. The most refined results seem to be contained in the papers [5, 7, 8], where, anyway, the basic formulas are valid only if z/b is positive. It is interesting to note that many of the quoted results (see, e.g., formula (8) of [5]) provide the same principal term that appears in the main result of this paper (see formula (1.2) below), even if the range of validity is not the same (for example, in formula (8) of [5], one must have $(b - a - 1)/z > 1$). Finally, among the most popular webpages dedicated to Kummer's function, very detailed information can be found at https://dlmf.nist.gov/13.8, where, in any case, the asymptotic results at issue are displayed under the above-mentioned restrictions. See also the printed version [6].

In this paper, the problem is solved without further restrictions on (the limiting value of) z/b, except that of being a negative number. Indeed, the above-mentioned restrictions are due to the method of asymptotic approximation of integrals, such as the well-known Laplace methods and its generalizations, and do not seem intrinsic characteristics of the problem itself. Therefore, the main statement of this note, whose proof is based on simple probabilistic arguments, shows that $M(a, b; z)$ converges to a distinguished limiting value whenever b and z approach $+\infty$ and $-\infty$, respectively, with the same order, up to an error of the same magnitude of $1/b$.

Proposition 1.1. *Let $a, \gamma, \lambda > 0$ and $c \geq a$ be fixed real numbers, and assume that $b = \gamma n + c$ and $z = -\lambda n$. Then, for any $n \in \mathbb{N}$, there holds*

$$\Big|M(a, \gamma n + c; -\lambda n) - \left(\frac{\gamma}{\gamma + \lambda}\right)^a \Big| \leq \frac{C(a, c, \gamma, \lambda)}{n} \tag{1.2}$$

with $C(a, c, \gamma, \lambda) = 16\gamma^{-1} + a(c+1)\lambda\gamma^{-2} + \frac{1}{2}(6\gamma + 3\gamma^2)^{1/2}K(a, \gamma, \lambda)$, where

$$K(a, \gamma, \lambda) := \left[2\lambda^4 a^2 (a+1)^2 \gamma^{-8}(1 + 2^8\gamma^{-8}) + 8\lambda^2 a^2 \gamma^{-6}(1 + 2^6\gamma^{-6})\right]^{1/2} .$$

As one can see, the generalization consists in the absence of any restriction on λ/γ.

2. Proof

The starting point for the present analysis is the following integral representation

$$M(a, \gamma n + c; -\lambda n) = \int_0^1 e^{-\lambda n t} B(a, \gamma n + (c - a); t) \mathrm{d}t \tag{2.1}$$

where $B(a, b; t) := \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} t^{a-1}(1-t)^{b-1}$ stands for the beta density with parameters a and b, with $t \in (0, 1)$. See formula (1) in Section VI.5 of [3]. To proceed, one invokes the well-known representation of the beta distribution in terms of a ratio of gamma random variables (see, e.g., Section 2 in Chapter 25 of [4]), according to which $B(a, \gamma n + (c - a); \cdot)$ can be seen as the density of the ratio $U/(U + V + S_n)$, where U, V and S_n are three independent random variables distributed as follows: $U \sim \mathfrak{G}(a, 1)$, $V \sim \mathfrak{G}(c - a, 1)$ and $S_n \sim \mathfrak{G}(\gamma n, 1)$. Here, $\mathfrak{G}(\alpha, \beta)$ denotes the gamma distribution with shape parameter $\alpha \geq 0$ and scale parameter $\beta > 0$, with the proviso that $\mathfrak{G}(0,\beta)$ coincides with the degenerate distribution at 0. Then, one defines $\psi_n(u,v;x) := \frac{\lambda n u}{u+v+x}$, $\phi_n(u,v;x) := e^{-\psi_n(u,v;x)}$ and

$$f_n(x) := \int_0^{+\infty}\int_0^{+\infty} \phi_n(u,v;x)\mathfrak{G}(a,1)(\mathrm{d}u)\mathfrak{G}(c-a,1)(\mathrm{d}v) = \mathsf{E}[\phi_n(U,V;x)] \quad (2.2)$$

where E denotes expectation. Since (2.1) can be translated as

$$M(a,\gamma n+c;-\lambda n) = \mathsf{E}\left[\phi_n(U,V;S_n)\right] = \mathsf{E}\big[\mathsf{E}[\phi_n(U,V;S_n) \mid S_n]\big] = \mathsf{E}[f_n(S_n)] \quad (2.3)$$

and $\mathsf{E}[S_n] = \gamma n$, one can resort to Taylor's formula with Lagrange remainder to obtain

$$f_n(S_n) = f_n(\gamma n) + f_n'(\gamma n)(S_n-\gamma n) + \frac{1}{2}f_n''(X_n)(S_n-\gamma n)^2$$

where X_n is a (random) point in the interval $[\min\{\gamma n, S_n\}, \max\{\gamma n, S_n\}]$. At this stage, thanks to (2.3), taking expectation in the last identity gives

$$M(a,\gamma n+c;-\lambda n) = f_n(\gamma n) + \frac{1}{2}\mathsf{E}\left[f_n''(X_n)(S_n-\gamma n)^2\right] \ . \quad (2.4)$$

Apropos of the term $f_n(\gamma n)$, one can exploit that $|e^{-x}-e^{-y}| \leq |x-y|$ holds for every $x,y>0$, to get

$$\begin{aligned} |\phi_n(u,v;\gamma n) - \exp\{-\lambda u/\gamma\}| &\leq \left|\psi_n(u,v;\gamma n) - \frac{\lambda u}{\gamma}\right| = \frac{\lambda u}{\gamma\left(\gamma+\frac{u+v}{n}\right)}\left(\frac{u+v}{n}\right) \\ &\leq \frac{\lambda u}{\gamma^2}\left(\frac{u+v}{n}\right) \ . \end{aligned}$$

Taking account of (2.2) and that $\mathsf{E}[\exp\{-\lambda U/\gamma\}]$ corresponds to the Laplace transform of U evaluated at λ/γ, which coincides with $\left(\frac{\gamma}{\gamma+\lambda}\right)^a$, one has

$$\left|f_n(\gamma n) - \left(\frac{\gamma}{\gamma+\lambda}\right)^a\right| \leq \mathsf{E}[U(U+V)]\frac{\lambda}{\gamma^2 n} = a(c+1)\frac{\lambda}{\gamma^2 n} \ . \quad (2.5)$$

It remains to bound $|M(a,b;z) - f_n(\gamma n)|$ by resorting to (2.4), leading to

$$\begin{aligned} |M(a,b;z) - f_n(\gamma n)| &\leq \frac{1}{2}\mathsf{E}\left[|f_n''(X_n)|(S_n-\gamma n)^2\right] \\ &\leq \frac{1}{2}\left(\mathsf{E}\left[|f_n''(X_n)|^2\right]\right)^{1/2}\left(\mathsf{E}\left[(S_n-\gamma n)^4\right]\right)^{1/2} \quad (2.6) \end{aligned}$$

by virtue of the Cauchy-Schwartz inequality. For the term $\left(\mathsf{E}\left[(S_n-\gamma n)^4\right]\right)^{1/2}$ one exploits an exact formula contained, for example, in [9], which gives

$$\mathsf{E}\left[(S_n-\gamma n)^4\right] = 6\gamma n + 3\gamma^2 n^2 \leq (6\gamma+3\gamma^2)n^2 \ . \quad (2.7)$$

Then, to study $\mathsf{E}\left[|f_n''(X_n)|^2\right]$, one starts by observing that

$$\frac{\partial^2}{\partial x^2}\phi_n(u,v;x) = e^{-\psi_n(u,v;x)}\left\{\left(\frac{\partial}{\partial x}\psi_n(u,v;x)\right)^2 - \frac{\partial^2}{\partial x^2}\psi_n(u,v;x)\right\}$$

holds true for all $x>0$. The first consequence of this identity is that, for all x in (x_1,x_2) with $0<x_1<x_2$, all the derivatives (with respect to x) of f_n can

be calculated by passing the derivative under the integral, thanks to a dominated convergence argument. Whence, for all $x > 0$,

$$\begin{aligned} f_n''(x) &= \mathsf{E}\left[e^{-\psi_n(U,V;x)}\left\{\left(\frac{\partial}{\partial x}\psi_n(U,V;x)\right)^2 - \frac{\partial^2}{\partial x^2}\psi_n(U,V;x)\right\}\right] \\ &= \mathsf{E}\left[e^{-\psi_n(U,V;x)}\left\{\frac{\lambda^2 n^2 U^2}{(U+V+x)^4} - \frac{2\lambda n U}{(U+V+x)^3}\right\}\right] \end{aligned}$$

holds, implying that

$$|f_n''(x)| \le \frac{\lambda^2 n^2 \mathsf{E}[U^2]}{x^4} + \frac{2\lambda n \mathsf{E}[U]}{x^3} = \frac{\lambda^2 a(a+1)n^2}{x^4} + \frac{2\lambda a n}{x^3} .$$

At this stage, considering the integers n for which $\gamma n \ge 16$ and recalling that $(1/X_n)^k \le (1/\gamma n)^k + (1/S_n)^k$ for any $k \in \mathbb{N}$, one gets

$$\begin{aligned} \mathsf{E}\left[|f_n''(X_n)|^2\right] &\le \frac{2\lambda^4 a^2(a+1)^2}{\gamma^8}\left(\frac{n^4}{n^8} + n^4\mathsf{E}\left[\frac{1}{S_n^8}\right]\right) + \frac{8\lambda^2 a^2}{\gamma^6}\left(\frac{n^2}{n^6} + n^2\mathsf{E}\left[\frac{1}{S_n^6}\right]\right) \\ &= \frac{2\lambda^4 a^2(a+1)^2}{\gamma^8}\left(\frac{1}{n^4} + n^4\frac{\Gamma(\gamma n-8)}{\Gamma(\gamma n)}\right) + \frac{8\lambda^2 a^2}{\gamma^6}\left(\frac{1}{n^4} + n^2\frac{\Gamma(\gamma n-6)}{\Gamma(\gamma n)}\right) \\ &\le \frac{2\lambda^4 a^2(a+1)^2}{\gamma^8}\left(\frac{1}{n^4} + \frac{2^8}{\gamma^8 n^4}\right) + \frac{8\lambda^2 a^2}{\gamma^6}\left(\frac{1}{n^4} + \frac{2^6}{\gamma^6 n^4}\right) \end{aligned}$$

yielding

$$\left(\mathsf{E}\left[|f_n''(X_n)|^2\right]\right)^{1/2} \le \frac{K(a,\gamma,\lambda)}{n^2} \tag{2.8}$$

with $K(a,\gamma,\lambda) = \left[2\lambda^4 a^2(a+1)^2\gamma^{-8}(1+2^8\gamma^{-8}) + 8\lambda^2 a^2\gamma^{-6}(1+2^6\gamma^{-6})\right]^{1/2}$. Then, under the restriction that $\gamma n \ge 16$, the proof of (1.2) is completed by combining (2.5)–(2.8). Finally, for $n < 16\gamma^{-1}$, it is enough to take cognizance that the the right-hand side of (1.2) is greater than 1, whilst left-hand side of (1.2) is in $(0,1)$. In particular, the last claim about the left-hand side follows from the fact that, for any $a, b, \lambda, \gamma > 0$ and $z < 0$, one has that both $M(a,b;z)$ and $[\gamma/(\gamma+\lambda)]^a$ belong to $(0,1)$.

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