RECENT OBSERVATIONS ON NONLINEAR TWO-PARAMETER SINGULAR INTEGRAL OPERATORS

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Abstract. In this paper, we present some theorems concerning weighted pointwise convergence of nonlinear singular integral operators of the form:

\[(T_\xi f)(x) = \int_a^b K_\xi (t-x, f(t)) \, dt, \quad x \in (a, b), \quad \xi \in \Xi,\]

where \((a, b)\) is a certain finite interval in \(\mathbb{R}\), \(\Xi\) is a non-empty set of indices and \(f\) is measurable function on \((a, b)\) in the sense of Lebesgue.

1. Introduction and Preliminaries

The singular integral operators, which are derived from the computation of the partial sums of the Fourier series of functions, play a vital role in the approximation theory. Approximation theory researchers are familiar with the following integral operators:

\[(L_\zeta f)(x) = \int_{-\pi}^\pi f(t) K_\zeta (t-x) \, dt, \quad x \in [-\pi, \pi], \quad \zeta \in \Omega,\]

where \(\Omega\) is a non-empty set of indices with accumulation point \(\zeta_0\) and \(K_\zeta, K_\zeta : \mathbb{R} \to \mathbb{R}_+\) for any \(\zeta \in \Omega\), denotes the kernel function satisfying some properties. Approximation properties of the operators of type \((1)\) in Lebesgue spaces and their generalizations were widely investigated in \([1], [5], [7], [8], [9], [15], [16]\) and \([18]\).

Also, some weighted approximation results concerning the operators of type \((1)\) and its generalizations can be found in \([1], [2], [6], [11]\) and \([19]\).

Let \(G\) be a locally compact Abelian group with the Haar measure. The following nonlinear singular integral operators

\[(T_w f) (y) = \int_G K_w(x-y, f(x)) dx, \quad y \in G, \quad w \in \Lambda,\]

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where $\Lambda$ is a non-empty set of indices with any topology and $w_0$ is an accumulation point of this set in this topology and $(K_w)_{w \in \Lambda}$ is a family of kernel functions $K_w : G \times \mathbb{R} \to \mathbb{R}$ which satisfy some properties including Lipschitz condition with respect to the second variable, were initially considered by Musielak [12]. The pointwise convergence of linear and two-parameter case of the operators of type (2) was studied first by Taberski [18]. Then, Swiderski and Wachnicki [17] proved pointwise convergence of the operators of type (2) which depend on two parameters at continuity points and Lebesgue points of the functions belonging to $L_p(G)$ ($1 \leq p \leq \infty$), where $G = \mathbb{R}$ or $G = [-\pi, \pi)$. Later on, Karslı [10] proved pointwise convergence of the operators of type (2) by considering them with two parameters and taking domain of integration as $(a, b)$ at $\mu$-generalized Lebesgue points of the functions belonging to $L_1(a, b)$, where $(a, b)$ is an arbitrary interval in $\mathbb{R}$. For detailed information concerning convergence properties of the operators of type (2) in different settings, we refer the reader to [3], [4], [13], [14] and [21].

In the current manuscript, weighted approximation by nonlinear singular integral operators in the following form:

$$\left( T_\xi f \right)(x) = \int_a^b K_\xi (t - x, f(t)) \, dt, \quad x \in (a, b), \quad \xi \in \Xi,$$

are taken under the spotlights, where $(a, b)$ is a certain, finite interval in $\mathbb{R}$ on which the real-valued function $f$ is measurable in the sense of Lebesgue and $\Xi$ is a non-empty set of indices with any topology, $\xi_0$ is an accumulation point of this set in this topology and $(K_\xi)_{\xi \in \Xi}$ is a family of kernel functions $K_\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying some properties including Lipschitz condition with respect to the second variable. Our main concern is to prove that the operators of type (3) converge to the functions $f \in L^1_{\varphi}(a, b)$ at some $\mu$-generalized Lebesgue points of these functions, where $\varphi$ is a certain weight function defined on $\mathbb{R}$, that is, $\varphi$ is conventionally required to be positive and measurable function on its domain in order to obtain a definition of the norm for the space $L^1_{\varphi}(a, b)$ consisting of the functions $f$ such that $\|f\|_{L^1_{\varphi}(a, b)} := \int_a^b \frac{|f(t)|}{\varphi(t)} \, dt < \infty$ (see, [11], [19]). This manuscript may be seen as a continuation of [17], [10], and a generalization of Theorem 3 for the case $\langle a, b \rangle = (a, b)$ in [20]. Although the space of measurable functions in the sense of Lebesgue contains integrable functions, our main concern is to investigate the conditions under which pointwise convergence exists for the functions which are measurable but are not integrable in the sense of Lebesgue as $(x, \xi)$ tends to $(x_0, \xi_0)$. On the other hand, we will follow slightly different approach, that is, we will set the conditions for the kernel function so that we do not need external theorems as in [20].

The paper is structured as follows: In Section 2, we prove main result concerning weighted pointwise convergence of the operators of type (3). In Section 3, we establish the rate of pointwise convergence.

We start by considering a point $x_0 \in (a, b)$ which stands for a $\mu$-generalized Lebesgue point of the function $f \in L^1_{\varphi}(a, b)$ at which

$$\lim_{h \to 0} \frac{1}{\mu(h)} \int_{x_0-h}^{x_0} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \, dt = 0 \quad (4)$$
for $h > 0$ satisfying $(x_0 - h, x_0 + h) \subseteq (a, b)$, hold. Here, the function $\mu$ is an increasing and absolutely continuous function on $[0, b - a]$ with $\mu(0) = 0$. The characterization of function $\mu$ was given by Gadjiev [8] in order to generalize Natanson type lemmas given in [18]. The properties of the function $\mu(h)$ were designated similar to the function $h$, and therefore taking $\mu(h) = h$ gave the classical definition of $d-$ point. Later on, $\mu-$generalized Lebesgue point or $\mu-$Lebesgue point notions were used and this function came to the mind. Then, $\mu-$generalized Lebesgue point notion has taken its place in the literature (for different usages, see, for example, [15, 10, 10, 7]).

Now, we consider a class of functions where our kernel functions belong to.

Let $\Xi$ be a non-empty set of indices with any topology and $\xi_0$ be an accumulation point of this set in this topology. Following the concept in [12, 4, 17], a family $(K_\xi)_{\xi \in \Xi}$ of functions $K_\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a kernel if $K_\xi(t, 0) = 0$ for all $t \in \mathbb{R}$ and $\xi \in \Xi$ and $K_\xi(t, u)$ is integrable in the sense of Lebesgue with respect to first variable for all values of the second variable. Also, we consider a function $\omega$, the definition of which is based on the weight function $\varphi > 0$ such that $\omega(y) := \sup_{w \in \mathbb{R}} \frac{\varphi(y+w)}{\varphi(w)}$ is finite for all $y \in \mathbb{R}$.

We suppose that at a point $x_0 \in (a, b)$ which stands for a $\mu-$generalized Lebesgue point of the function $f \in L^1_{\cal{R}}(a, b)$, a family $(K_\xi)_{\xi \in \Xi}$ of functions $K_\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

a) $K_\xi(t, 0) = 0$ for all $t \in \mathbb{R}$ and $\xi \in \Xi$ and $K_\xi(t, u)$ is integrable in the sense of Lebesgue with respect to first variable for all values of the second variable.

b) There exists a function $L_\xi \geq 0$ for all $\xi \in \Xi$ which is integrable over $\mathbb{R}$ for each $\xi \in \Xi$ in the sense of Lebesgue such that the following inequality:

$$|K_\xi(t, u) - K_\xi(t, v)| \leq L_\xi(t) |u - v|$$

holds for all $t, u, v \in \mathbb{R}$ and for any $\xi \in \Xi$.

c) $\lim_{(x, \xi) \to (x_0, \xi_0)} \int_{-\infty}^{\infty} K_\xi(t - x, \frac{u}{\varphi(x_0)} \varphi(t))dt - u = 0$, for all $u \in \mathbb{R}$.

d) $\lim_{(x, \xi) \to (x_0, \xi_0)} \omega(x) \sup_{t \in \mathbb{R} \setminus (-\xi, \xi)} (\varphi(t)L_\xi(t)) = 0$, for every $\zeta > 0$.

e) $\lim_{(x, \xi) \to (x_0, \xi_0)} \omega(x) \int_{t \in \mathbb{R} \setminus (-\xi, \xi)} \varphi(t)L_\xi(t)dt = 0$, for every $\zeta > 0$.

f) $\varphi(t)L_\xi(t)$ is non-decreasing function as a function of $t$ on $(-\infty, 0]$ and non-increasing function as a function of $t$ on $[0, \infty)$ for any $\xi \in \Xi$.

g) $\|\omega L_\xi\|_{L_1(\mathbb{R})} \leq M < \infty$, for all $\xi \in \Xi$, where $M > 0$ is a certain real number.

Throughout this manuscript, we assume that all of the above conditions hold.

**Remark.** In the above conditions, we imposed the conditions (d) – (g) to the function $\varphi(t)L_\xi(t)$ instead of $L_\xi(t)$ as in usual $L^1(a, b)$ case. The conditions (a) – (b) and (g) are compulsory in order to justify well-definiteness of the operators, that
is $T_\xi f \in L^1_\phi(a,b)$ whenever $f \in L^1_\phi(a,b)$ (see, [20]). For the conditions which are related to above ones, we refer the reader to [3], [6], [2], [7], [8], [9], [10], and [17].

2. Main Result

Theorem 2.1. If $x_0 \in (a,b)$ is a $\mu$-generalized Lebesgue point of the function $f \in L^1_\phi(a,b)$, then

$$\lim_{(x,\xi) \to (x_0,\xi_0)} |(T_\xi f)(x) - f(x_0)| = 0$$

on any set $Z$ on which the function

$$B_\delta(x,\xi) := \omega(x) \left( \int_{x_0+\delta}^{x_0-\delta} \phi(t-x)L_\xi(t-x) \left| \{ \mu \left( |t-x_0| \right) \right\}_t \, dt + 2L_\xi(0)\phi(0)\mu \left( |x-x_0| \right) \right)$$

for a certain real number $\delta^*$ satisfying $0 < \delta \leq \delta^*$ and $(x_0 - \delta^*, x_0 + \delta^*) \subseteq (a,b)$, remains bounded as $(x,\xi)$ tends to $(x_0,\xi_0)$.

Proof. Set $I = |(T_\xi f)(x) - f(x_0)|$. We may define a function $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$g(u) := \begin{cases} f(u), & u \in (a,b) \\ 0, & \text{otherwise.} \end{cases}$$

Now, in view of condition (c), we may write

$$I = \left| \int_a^b K_\xi(t-x, f(t)) dt - f(x_0) \right|$$

$$= \left| \int_{-\infty}^{\infty} K_\xi(t-x, g(t)) dt - f(x_0) \right|$$

$$= \left| \int_{-\infty}^{\infty} K_\xi(t-x, g(t)) dt - \int_{-\infty}^{\infty} K_\xi(t-x, f(x_0)\phi(t)) dt \\
+ \int_{-\infty}^{\infty} K_\xi(t-x, f(x_0)\phi(t)) dt - f(x_0) \right|.$$
is obtained. In view of condition (c), \( I_2 \to 0 \) as \((x, \xi)\) tends to \((x_0, \xi_0)\). Now, using change of variables and properties of \( \varphi \) for \( I_1 \) we get:

\[
I_1 = \int_{-\infty}^{\infty} \frac{g(t)}{\varphi(t)} \varphi(t) L_{\xi}(t-x) dt
\]

\[
= \int_{-\infty}^{\infty} \frac{g(s+x)}{\varphi(s+x)} \varphi(s+x) L_{\xi}(s) ds
\]

\[
= \int_{-\infty}^{\infty} \frac{g(s+x)}{\varphi(s+x)} \frac{f(x_0)}{\varphi(x_0)} \varphi(s+x) L_{\xi}(s) ds
\]

\[
\leq \omega(x) \int_{-\infty}^{\infty} \frac{g(s+x)}{\varphi(s+x)} \left| \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(s) L_{\xi}(s) ds
\]

\[
= \omega(x) I_{11}.
\]

Let \( \frac{a+b}{2} \leq x_0 < b \). In view of limit relations (4) and (5), for all \( \varepsilon > 0 \) there exists \( 0 < \delta \leq \delta' \) such that

\[
\int_{x_0-h}^{x_0} \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} dt \leq \varepsilon \mu(h)
\]

\[
\int_{x_0}^{x_0+h} \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} dt \leq \varepsilon \mu(h)
\]

hold provided that \( 0 < h \leq \delta \), respectively. Suppose that \( \delta < b - x_0 \) and \( 0 < x_0 - x < \frac{\delta}{2} \). The following inequality holds for \( I_{11} \):

\[
I_{11} = \left\{ \int_{(x_0-h, x_0+\delta)} + \int_{\mathbb{R} \setminus (x_0-h, x_0+\delta)} \right\} \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \varphi(t-x) L_{\xi}(t-x) dt
\]

\[
\leq \int_{(x_0-h, x_0+\delta)} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(t-x) L_{\xi}(t-x) dt
\]

\[
+ \int_{\mathbb{R} \setminus (x_0-h, x_0+\delta)} \left| \frac{g(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(t-x) L_{\xi}(t-x) dt
\]

\[
= (I_{111} + I_{112}).
\]
Next, we consider the integral $I_{111}$. Splitting $I_{111}$ into two parts yields:

$$I_{111} = \left\{ \int_{x_0}^{x_0+\delta} + \int_{x_0-\delta}^{x_0} \right\} \frac{f(t)}{\varphi(t)} \left( \frac{f(x_0)}{\varphi(x_0)} \right) \varphi(t-x)L_\xi(t-x)dt$$

$$= I_{1111} + I_{1112}.$$

Now, as in [8][15][7], in order to simplify the proof we apply Lemma 2 in [8] to $I_{1111}$ and Lemma 1 in [8] to $I_{1112}$. Therefore, we obtain

$$I_{111} \leq \epsilon \left\{ \int_{x_0-\delta}^{x_0+\delta} \varphi(t-x)L_\xi(t-x) \left| \mu \right| dt + 2\varphi(0)L_\xi(0)\mu \left| x - x_0 \right| \right\}.$$

Now, we consider $I_{112}$. By the assumption $0 < x_0 - x < \frac{\delta}{2}$, we obtain

$$I_{112} = \int_{\mathbb{R} \setminus (x_0-\delta, x_0+\delta)} \frac{g(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \varphi(t-x)L_\xi(t-x)dt$$

$$\leq \int_{\mathbb{R} \setminus (x_0-\delta, x_0+\delta)} \frac{g(t)}{\varphi(t)} \varphi(t-x)L_\xi(t-x)dt$$

$$+ \int_{\mathbb{R} \setminus (x_0-\delta, x_0+\delta)} \frac{f(x_0)}{\varphi(x_0)} \varphi(t-x)L_\xi(t-x)dt$$

$$\leq \int_{\mathbb{R} \setminus \left( \frac{a+b}{2}, \frac{a+b}{2} \right)} \frac{g(s+x)}{\varphi(s+x)} \varphi(s)L_\xi(s)ds$$

$$+ \int_{\mathbb{R} \setminus \left( \frac{a+b}{2}, \frac{a+b}{2} \right)} \frac{f(x_0)}{\varphi(x_0)} \varphi(t-x)L_\xi(t-x)dt.$$

Hence, we can write

$$I_{112} \leq \|f\|_{L^1_{\xi}(a,b)} \sup_{s \in \mathbb{R} \setminus \left( \frac{a+b}{2}, \frac{a+b}{2} \right)} \left| \varphi(s)L_\xi(s) \right| + \left| \frac{f(x_0)}{\varphi(x_0)} \right| \int_{\mathbb{R} \setminus \left( \frac{a+b}{2}, \frac{a+b}{2} \right)} \varphi(s)L_\xi(s)ds.$$

Hence, the following inequality holds:

$$I_1 \leq \omega(x) \|f\|_{L^1_{\xi}(a,b)} \sup_{s \in \mathbb{R} \setminus \left( \frac{a+b}{2}, \frac{a+b}{2} \right)} \left| \varphi(s)L_\xi(s) \right| + \epsilon B_2(x, \xi)$$

$$+ \omega(x) \left| \frac{f(x_0)}{\varphi(x_0)} \right| \int_{\mathbb{R} \setminus \left( \frac{a+b}{2}, \frac{a+b}{2} \right)} \varphi(s)L_\xi(s)ds.$$

Therefore, if the points $(x, \xi) \in Z$ are sufficiently close to $(x_0, \xi_0)$, then the second term in the above sum tends to zero by the hypothesis. Also, remaining two terms tend to zero by conditions (d) and (e), respectively. Following similar steps, the same result is obtained for the case $a < x_0 \leq \frac{a+b}{2}$. Hence, the proof is completed. \(\Box\)
3. Rate of Convergence

**Theorem 3.1.** Suppose that the hypotheses of Theorem 2.1 are satisfied and the following conditions hold there:

(i) \( B_\delta(x, \xi) \to 0 \) as \((x, \xi) \to (x_0, \xi_0)\) for some \(0 < \delta \leq \delta^*\).

(ii) For every \( \zeta > 0 \), we have

\[
\omega(x) \sup_{t \in \mathbb{R} \setminus (-\zeta, \zeta)} \varphi(t) L_\xi(t) = o(B_\delta(x, \xi))
\]

as \((x, \xi) \to (x_0, \xi_0)\).

(iii) For every \( \zeta > 0 \), we have

\[
\omega(x) \int_{t \in \mathbb{R} \setminus (-\zeta, \zeta)} \varphi(t) L_\xi(t) dt = o(B_\delta(x, \xi))
\]

as \((x, \xi) \to (x_0, \xi_0)\).

(iv) Letting \((x, \xi) \to (x_0, \xi_0)\), we have

\[
\left| \int \int K_\xi(t - x, \frac{f(x_0)}{\varphi(x_0)} \varphi(t)) dt - f(x_0) \right| = o(B_\delta(x, \xi)).
\]

Then, at each \( \mu \)-generalized Lebesgue point of \( f \in L^1_\varphi(a, b) \), we have

\[
|(T_\xi f)(x) - f(x_0)| = o(B_\delta(x, \xi))
\]

as \((x, \xi) \to (x_0, \xi_0)\).

**Proof.** Under the hypotheses of Theorem 2.1, we have

\[
|(T_\xi f)(x) - f(x_0)| \leq \omega(x) \| f \|_{L^1_\varphi(a, b)} \sup_{s \in \mathbb{R} \setminus (-\frac{1}{2}, \frac{1}{2})} [\varphi(s) L_\xi(s)] + \varepsilon B_\delta(x, \xi)
\]

\[
+ \omega(x) \left| \frac{f(x_0)}{\varphi(x_0)} \right| \int_{s \in \mathbb{R} \setminus (-\frac{1}{2}, \frac{1}{2})} \varphi(s) L_\xi(s) ds
\]

\[
+ \left| \int \int K_\xi(t - x, \frac{f(x_0)}{\varphi(x_0)} \varphi(t)) dt - f(x_0) \right|.
\]

The remaining part is obvious by (i) – (iv). This completes the proof. \(\square\)

4. Conclusion

In this manuscript, weighted pointwise convergence theorem is presented for the operators of type (3) under certain conditions in finite domain and rate of convergence is established with respect to this main result.

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