Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 10 Issue 2(2019), Pages 1-9.

RECENT OBSERVATIONS ON NONLINEAR TWO-PARAMETER SINGULAR INTEGRAL OPERATORS

DIPTI TAPIAWALA, GUMRAH UYSAL, VISHNU NARAYAN MISHRA*

Abstract. In this paper, we present some theorems concerning weighted pointwise convergence of nonlinear singular integral operators of the form:

$$(T_\xi f)(x) = \int_a^b K_\xi\left(t-x, f\left(t\right)\right) dt, \quad x \in (a,b)\,,\ \xi \in \Xi,$$

where (a,b) is a certain finite interval in $\mathbb{R}$, Ξ is a non-empty set of indices and f is measurable function on (a,b) in the sense of Lebesgue.

1. Introduction and Preliminaries

The singular integral operators, which are derived from the computation of the partial sums of the Fourier series of functions, play a vital role in the approximation theory. Approximation theory researchers are familiar with the following integral operators:

$$\left(L_\zeta f\right)(x) = \int_{-\pi}^{\pi} f\left(t\right) K_\zeta\left(t-x\right) dt, \quad x \in [-\pi,\pi]\,, \quad \zeta \in \Omega, \tag{1}$$

where Ω is a non-empty set of indices with accumulation point ζ_0 and K_ζ, $K_\zeta : \mathbb{R} \to \mathbb{R}_0^+$ for any $\zeta \in \Omega$, denotes the kernel function satisfying some properties. Approximation properties of the operators of type (1) in Lebesgue spaces and their generalizations were widely investigated in [1], [5], [7], [8], [9], [15], [16] and [18]. Also, some weighted approximation results concerning the operators of type (1) and its generalizations can be found in [1], [2], [6], [11] and [19].

Let $\mathbf{G}$ be a locally compact Abelian group with the Haar measure. The following nonlinear singular integral operators

$$\left(T_w f\right)(y) = \int_{\mathbf{G}} K_w(x-y, f(x))dx,\ y \in \mathbf{G},\ w \in \Lambda, \tag{2}$$

2010 *Mathematics Subject Classification.* 41A35, 41A25, 28C10, 45P05.

Key words and phrases. $\mu-$generalized Lebesgue point; pointwise convergence; weighted approximation; nonlinear integral operators.

Submitted August 29, 2018. Published February 24, 2019.

* Corresponding author.

Communicated by Serkan Araci.

where Λ is a non-empty set of indices with any topology and w_0 is an accumulation point of this set in this topology and $(K_w)_{w\in\Lambda}$ is a family of kernel functions $K_w : \mathbf{G}\times\mathbb{R} \to \mathbb{R}$ which satisfy some properties including Lipschitz condition with respect to the second variable, were initially considered by Musielak [12]. The pointwise convergence of linear and two-parameter case of the operators of type (2) was studied first by Taberski [18]. Then, Swiderski and Wachnicki [17] proved pointwise convergence of the operators of type (2) which depend on two parameters at continuity points and Lebesgue points of the functions belonging to $L_p(G)$ $(1 \leq p \leq \infty)$, where $G = \mathbb{R}$ or $G = [-\pi, \pi)$. Later on, Karslı [10] proved pointwise convergence of the operators of type (2) by considering them with two parameters and taking domain of integration as $\langle a, b\rangle$ at $\mu-$generalized Lebesgue points of the functions belonging to $L_1 \langle a, b\rangle$, where $\langle a, b\rangle$ is an arbitrary interval in $\mathbb{R}$. For detailed information concerning convergence properties of the operators of type (2) in different settings, we refer the reader to [3], [4], [13], [14] and [21].

In the current manuscript, weighted approximation by nonlinear singular integral operators in the following form:

$$(T_\xi f)(x) = \int_a^b K_\xi \left(t - x, f\left(t\right)\right) dt, \quad x \in (a, b), \quad \xi \in \Xi, \tag{3}$$

are taken under the spotlights, where (a, b) is a certain, finite interval in $\mathbb{R}$ on which the real-valued function f is measurable in the sense of Lebesgue and Ξ is a non-empty set of indices with any topology, ξ_0 is an accumulation point of this set in this topology and $(K_\xi)_{\xi\in\Xi}$ is a family of kernel functions $K_\xi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfying some properties including Lipschitz condition with respect to the second variable. Our main concern is to prove that the operators of type (3) converge to the functions $f \in L^1_\varphi(a, b)$ at some $\mu-$generalized Lebesgue points of these functions, where φ is a certain weight function defined on $\mathbb{R}$, that is, φ is conventionally required to be positive and measurable function on its domain in order to obtain a definition of the norm for the space $L^1_\varphi(a, b)$ consisting of the functions f such that $\|f\|_{L^1_\varphi(a,b)} := \int_a^b \frac{|f(t)|}{\varphi(t)} dt < \infty$ (see, [11],[19]). This manuscript may be seen as a continuation of [17], [10], and a generalization of Theorem 3 for the case $\langle a, b\rangle = (a, b)$ in [20]. Although the space of measurable functions in the sense of Lebesgue contains integrable functions, our main concern is to investigate the conditions under which pointwise convergence exists for the functions which are measurable but are not integrable in the sense of Lebesgue as (x, ξ) tends to (x_0, ξ_0). On the other hand, we will follow slightly different approach, that is, we will set the conditions for the kernel function so that we do not need external theorems as in [20].

The paper is structured as follows: In Section 2, we prove main result concerning weighted pointwise convergence of the operators of type (3). In Section 3, we establish the rate of pointwise convergence.

We start by considering a point $x_0 \in (a, b)$ which stands for a $\mu-$generalized Lebesgue point of the function $f \in L^1_\varphi(a, b)$ at which

$$\lim_{h\to 0} \frac{1}{\mu(h)} \int_{x_0-h}^{x_0} \left| \frac{f\left(t\right)}{\varphi\left(t\right)} - \frac{f(x_0)}{\varphi\left(x_0\right)} \right| dt = 0 \tag{4}$$

$$\lim_{h\to 0}\frac{1}{\mu(h)}\int_{x_0}^{x_0+h}\left|\frac{f(t)}{\varphi(t)}-\frac{f(x_0)}{\varphi(x_0)}\right|dt=0 \tag{5}$$

for $h>0$ satisfying $(x_0-h,x_0+h)\subseteq(a,b)$, hold. Here, the function μ is an increasing and absolutely continuous function on $[0,b-a]$ with $\mu(0)=0$. The characterization of function μ was given by Gadjiev [8] in order to generalize Natanson type lemmas given in [18]. The properties of the function $\mu(h)$ were designated similar to the function h, and therefore taking $\mu(h)=h$ gave the classical definition of $d-$ point. Later on, $\mu-$generalized Lebesgue point or $\mu-$Lebesgue point notions were used and this function came to the mind. Then, $\mu-$generalized Lebesgue point notion has taken its place in the literature (for different usages, see, for example, [15, 16, 10, 7]).

Now, we consider a class of functions where our kernel functions belong to.

Let Ξ be a non-empty set of indices with any topology and ξ_0 be an accumulation point of this set in this topology. Following the concept in [12, 4, 17], a family $(K_\xi)_{\xi\in\Xi}$ of functions $K_\xi:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is called a kernel if $K_\xi(t,0)=0$ for all $t\in\mathbb{R}$ and $\xi\in\Xi$ and $K_\xi(t,u)$ is integrable in the sense of Lebesgue with respect to first variable for all values of the second variable. Also, we consider a function ω, the definition of which is based on the weight function $\varphi>0$ such that $\omega(y):=\sup\limits_{w\in\mathbb{R}}\frac{\varphi(y+w)}{\varphi(w)}$ is finite for all $y\in\mathbb{R}$.

We suppose that at a point $x_0\in(a,b)$ which stands for a $\mu-$generalized Lebesgue point of the function $f\in L^1_\varphi(a,b)$, a family $(K_\xi)_{\xi\in\Xi}$ of functions $K_\xi:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies the following conditions:

a) $K_\xi(t,0)=0$ for all $t\in\mathbb{R}$ and $\xi\in\Xi$ and $K_\xi(t,u)$ is integrable in the sense of Lebesgue with respect to first variable for all values of the second variable.

b) There exists a function $L_\xi\geq 0$ for all $\xi\in\Xi$ which is integrable over $\mathbb{R}$ for each $\xi\in\Xi$ in the sense of Lebesgue such that the following inequality:

$$|K_\xi(t,u)-K_\xi(t,v)|\leq L_\xi(t)\,|u-v|$$

holds for all $t,u,v\in\mathbb{R}$ and for any $\xi\in\Xi$.

c) $\lim\limits_{(x,\xi)\to(x_0,\xi_0)}\left|\int\limits_{-\infty}^{\infty}K_\xi(t-x,\frac{u}{\varphi(x_0)}\varphi(t))dt-u\right|=0,$ for all $u\in\mathbb{R}$.

d) $\lim\limits_{(x,\xi)\to(x_0,\xi_0)}\left[\omega(x)\sup\limits_{t\in\mathbb{R}\setminus(-\zeta,\zeta)}(\varphi(t)L_\xi(t))\right]=0,$ for every $\zeta>0$.

e) $\lim\limits_{(x,\xi)\to(x_0,\xi_0)}\left[\omega(x)\int\limits_{t\in\mathbb{R}\setminus(-\zeta,\zeta)}\varphi(t)L_\xi(t)dt\right]=0,$ for every $\zeta>0$.

f) $\varphi(t)L_\xi(t)$ is non-decreasing function as a function of t on $(-\infty,0]$ and non-increasing function as a function of t on $[0,\infty)$ for any $\xi\in\Xi$.

g) $\|\omega L_\xi\|_{L_1(\mathbb{R})}\leq M<\infty,$ for all $\xi\in\Xi$, where $M>0$ is a certain real number.

Throughout this manuscript, we assume that all of the above conditions hold.

Remark. *In the above conditions, we imposed the conditions $(d)-(g)$ to the function $\varphi(t)L_\xi(t)$ instead of $L_\xi(t)$ as in usual $L^1(a,b)$ case. The conditions $(a)-(b)$ and (g) are compulsory in order to justify well-definiteness of the operators, that*

is $T_\xi f \in L^1_\varphi(a,b)$ *whenever* $f \in L^1_\varphi(a,b)$ *(see,* [20]*). For the conditions which are related to above ones, we refer the reader to* [3], [6], [2], [7], [8], [9], [10], [11], [17] *and* [20].

2. Main Result

Theorem 2.1. *If* $x_0 \in (a,b)$ *is a* $\mu-$*generalized Lebesgue point of the function* $f \in L^1_\varphi(a,b)$, *then*

$$\lim_{(x,\xi)\to(x_0,\xi_0)} |(T_\xi f)(x) - f(x_0)| = 0$$

on any set Z *on which the function*

$$B_\delta(x,\xi) :=$$

$$\omega(x)\left(\int_{x_0-\delta}^{x_0+\delta} \varphi(t-x)L_\xi(t-x)\left|\{\mu(|t-x_0|)\}'_t\right| dt + 2L_\xi(0)\varphi(0)\mu(|x-x_0|)\right)$$

for a certain real number δ^* *satisfying* $0 < \delta \le \delta^*$ *and* $(x_0-\delta^*, x_0+\delta^*) \subseteq (a,b)$, *remains bounded as* (x,ξ) *tends to* (x_0,ξ_0).

Proof. Set $I = |(T_\xi f)(x) - f(x_0)|$. We may define a function $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$g(u) := \begin{cases} f(u), & u \in (a,b) \\ 0, & otherwise. \end{cases}$$

Now, in view of condition (c), we may write

$$\begin{aligned} I &= \left|\int_a^b K_\xi(t-x, f(t))dt - f(x_0)\right| \\ &= \left|\int_{-\infty}^{\infty} K_\xi(t-x, g(t))dt - f(x_0)\right| \end{aligned}$$

$$\begin{aligned} = &\left|\int_{-\infty}^{\infty} K_\xi(t-x, g(t))dt - \int_{-\infty}^{\infty} K_\xi(t-x, \frac{f(x_0)}{\varphi(x_0)}\varphi(t))dt \right. \\ &\left. + \int_{-\infty}^{\infty} K_\xi(t-x, \frac{f(x_0)}{\varphi(x_0)}\varphi(t))dt - f(x_0)\right|. \end{aligned}$$

Using condition (b), the following inequality:

$$\begin{aligned} I &\le \int_{-\infty}^{\infty} \left|\frac{g(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)}\right| \varphi(t)L_\xi(t-x)dt \\ &\quad + \left|\int_{-\infty}^{\infty} K_\xi(t-x, \frac{f(x_0)}{\varphi(x_0)}\varphi(t))dt - f(x_0)\right| \\ &= I_1 + I_2 \end{aligned}$$

is obtained. In view of condition (c), $I_2 \to 0$ as (x,ξ) tends to (x_0,ξ_0). Now, using change of variables and properties of φ for I_1 we get:

$$I_1 = \int_{-\infty}^{\infty} \left| \frac{g(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(t) L_\xi(t-x)dt$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \left| \frac{g(s+x)}{\varphi(s+x)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(s+x) L_\xi(s)ds \\
&= \int_{-\infty}^{\infty} \left| \frac{g(s+x)}{\varphi(s+x)} - \frac{f(x_0)}{\varphi(x_0)} \right| \frac{\varphi(s+x)}{\varphi(s)} \varphi(s) L_\xi(s)ds \\
&\leq \omega(x) \int_{-\infty}^{\infty} \left| \frac{g(s+x)}{\varphi(s+x)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(s) L_\xi(s)ds \\
&= \omega(x) \int_{-\infty}^{\infty} \left| \frac{g(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(t-x) L_\xi(t-x)dt \\
&= \omega(x) I_{11}.
\end{aligned}$$

Let $\frac{a+b}{2} \leq x_0 < b$. In view of limit relations (4) and (5), for all $\varepsilon > 0$ there exists $0 < \delta \leq \delta^*$ such that

$$\int_{x_0-h}^{x_0} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| dt \leq \varepsilon \mu(h)$$

$$\int_{x_0}^{x_0+h} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| dt \leq \varepsilon \mu(h)$$

hold provided that $0 < h \leq \delta$, respectively. Suppose that $\delta < b - x_0$ and $0 < x_0 - x < \frac{\delta}{2}$. The following inequality holds for I_{11}:

$$\begin{aligned}
I_{11} &= \left\{ \int_{(x_0-\delta,x_0+\delta)} + \int_{\mathbb{R}\setminus(x_0-\delta,x_0+\delta)} \right\} \left| \frac{g(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(t-x) L_\xi(t-x)dt \\
&\leq \int_{(x_0-\delta,x_0+\delta)} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(t-x) L_\xi(t-x)dt \\
&\quad + \int_{\mathbb{R}\setminus(x_0-\delta,x_0+\delta)} \left| \frac{g(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(t-x) L_\xi(t-x)dt \\
&= (I_{111} + I_{112}).
\end{aligned}$$

Next, we consider the integral I_{111}. Splitting I_{111} into two parts yields:

$$\begin{aligned} I_{111} &= \left\{ \int_{x_0-\delta}^{x_0} + \int_{x_0}^{x_0+\delta} \right\} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(t-x) L_\xi(t-x) dt \\ &= I_{1111} + I_{1112}. \end{aligned}$$

Now, as in [8, 15, 7], in order to simplify the proof we apply Lemma 2 in [8] to I_{1111} and Lemma 1 in [8] to I_{1112}. Therefore, we obtain

$$I_{111} \leq \varepsilon \left\{ \int_{x_0-\delta}^{x_0+\delta} \varphi(t-x) L_\xi(t-x) \left| \{\mu |t-x_0|\}'_t \right| dt + 2\varphi(0) L_\xi(0) \mu(|x-x_0|) \right\}.$$

Now, we consider I_{112}. By the assumption $0 < x_0 - x < \frac{\delta}{2}$, we obtain

$$\begin{aligned} I_{112} &= \int_{\mathbb{R}\setminus(x_0-\delta, x_0+\delta)} \left| \frac{g(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(t-x) L_\xi(t-x) dt \\ &\leq \int_{\mathbb{R}\setminus(x_0-\delta, x_0+\delta)} \left| \frac{g(t)}{\varphi(t)} \right| \varphi(t-x) L_\xi(t-x) dt \\ &\quad + \left| \frac{f(x_0)}{\varphi(x_0)} \right| \int_{\mathbb{R}\setminus(x_0-\delta, x_0+\delta)} \varphi(t-x) L_\xi(t-x) dt \\ &\leq \int_{\mathbb{R}\setminus\left(\frac{-\delta}{2}, \frac{\delta}{2}\right)} \left| \frac{g(s+x)}{\varphi(s+x)} \right| \varphi(s) L_\xi(s) ds \\ &\quad + \left| \frac{f(x_0)}{\varphi(x_0)} \right| \int_{\mathbb{R}\setminus\left(\frac{-\delta}{2}, \frac{\delta}{2}\right)} \varphi(t-x) L_\xi(t-x) dt. \end{aligned}$$

Hence, we can write

$$I_{112} \leq \|f\|_{L^1_\varphi(a,b)} \sup_{s\in\mathbb{R}\setminus\left(\frac{-\delta}{2}, \frac{\delta}{2}\right)} [\varphi(s) L_\xi(s)] + \left| \frac{f(x_0)}{\varphi(x_0)} \right| \int_{s\in\mathbb{R}\setminus\left(\frac{-\delta}{2}, \frac{\delta}{2}\right)} \varphi(s) L_\xi(s) ds.$$

Hence, the following inequality holds:

$$\begin{aligned} I_1 &\leq \omega(x) \|f\|_{L^1_\varphi(a,b)} \sup_{s\in\mathbb{R}\setminus\left(\frac{-\delta}{2}, \frac{\delta}{2}\right)} [\varphi(s) L_\xi(s)] + \varepsilon B_\delta(x,\xi) \\ &\quad + \omega(x) \left| \frac{f(x_0)}{\varphi(x_0)} \right| \int_{s\in\mathbb{R}\setminus\left(\frac{-\delta}{2}, \frac{\delta}{2}\right)} \varphi(s) L_\xi(s) ds. \end{aligned}$$

Therefore, if the points $(x,\xi) \in Z$ are sufficiently close to (x_0, ξ_0), then the second term in the above sum tends to zero by the hypothesis. Also, remaining two terms tend to zero by conditions (d) and (e), respectively. Following similar steps, the same result is obtained for the case $a < x_0 \leq \frac{a+b}{2}$. Hence, the proof is completed. □

3. Rate of Convergence

Theorem 3.1. *Suppose that the hypotheses of Theorem 2.1 are satisfied and the following conditions hold there:*

(*i*) $B_\delta(x,\xi) \to 0$ as $(x,\xi) \to (x_0,\xi_0)$ for some $0 < \delta \leq \delta^*$.

(*ii*) For every $\zeta > 0$, we have

$$\omega(x) \sup_{t\in\mathbb{R}\setminus(-\zeta,\zeta)} \varphi(t) L_\xi(t) = \mathbf{o}\left(B_\delta(x,\xi)\right)$$

as $(x,\xi) \to (x_0,\xi_0)$.

(*iii*) For every $\zeta > 0$, we have

$$\omega(x) \int\limits_{t\in\mathbb{R}\setminus(-\zeta,\zeta)} \varphi(t) L_\xi(t) dt = \mathbf{o}\left(B_\delta(x,\xi)\right)$$

as $(x,\xi) \to (x_0,\xi_0)$.

(*iv*) Letting $(x,\xi) \to (x_0,\xi_0)$, we have

$$\left| \int\limits_{-\infty}^{\infty} K_\xi(t-x, \frac{f(x_0)}{\varphi(x_0)}\varphi(t)) dt - f(x_0) \right| = \mathbf{o}\left(B_\delta(x,\xi)\right).$$

Then, at each $\mu-$generalized Lebesgue point of $f \in L^1_\varphi(a,b)$, we have

$$\left|\left(T_\xi f\right)(x) - f(x_0)\right| = \mathbf{o}\left(B_\delta(x,\xi)\right)$$

as $(x,\xi) \to (x_0,\xi_0)$.

Proof. Under the hypotheses of Theorem 2.1, we have

$$\begin{aligned} \left|\left(T_\xi f\right)(x) - f(x_0)\right| \quad \leq \quad & \omega(x) \|f\|_{L^1_\varphi(a,b)} \sup_{s\in\mathbb{R}\setminus\left(\frac{-\delta}{2},\frac{\delta}{2}\right)} \left[\varphi(s) L_\xi(s)\right] + \varepsilon B_\delta(x,\xi) \\ & + \omega(x) \left|\frac{f(x_0)}{\varphi(x_0)}\right| \int\limits_{s\in\mathbb{R}\setminus\left(\frac{-\delta}{2},\frac{\delta}{2}\right)} \varphi(s) L_\xi(s) ds \\ & + \left| \int\limits_{-\infty}^{\infty} K_\xi(t-x, \frac{f(x_0)}{\varphi(x_0)}\varphi(t)) dt - f(x_0) \right|. \end{aligned}$$

The remainig part is obvious by $(i)-(iv)$. This completes the proof. □

4. Conclusion

In this manuscript, weighted pointwise convergence theorem is presented for the operators of type (3) under certain conditions in finite domain and rate of convergence is established with respect to this main result.

5. Acknowledgements

The authors thank to anonymous referee for his/her valuable comments and suggestions which improve the flow of the manuscript.

References

[1] G. Alexits, Convergence Problems of Orthogonal Series, Translated from the German by I. Földer, International Series of Monographs in Pure and Applied Mathematics vol. 20, Pergamon Press, New York (1961).

[2] S.E. Almali, On approximation properties of non-convolution type of integral operators for non-integrable function, Adv. Math. Sci. J., **5**(2) (2016) 219–230.

[3] C. Bardaro, J. Musielak and G. Vinti, Approximation by nonlinear singular integral operators in some modular function spaces, Ann. Polon. Math., **63**(2) (1996) 173–182.

[4] C. Bardaro, J. Musielak and G. Vinti, Nonlinear Integral Operators and Applications, De Gruyter Ser. Nonlinear Anal. Appl. 9, Walter de Gruyter, Berlin (2003).

[5] P.L. Butzer and R.J. Nessel, Fourier Analysis and Approximation Vol. 1: One-dimensional theory, Pure and Applied Mathematics, Vol. 40, Academic Press, New York, London (1971).

[6] A.S. Džafarov, On the degree of approximation of functions by a family of linear integral operators, Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk, **3** (1962) 3–17.

[7] S. Esen, The order of approximation by the family of integral operators with positive kernel, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., **28** (2008) 117–122.

[8] A.D. Gadjiev, The order of convergence of singular integrals which depend on two parameters, Special Problems of Functional Analysis and their Appl. to the Theory of Diff. Eq. and the Theory of Func., Izdat. Akad. Nauk Azerbaĭdžan. SSR., (1968) 40–44.

[9] A.D. Gadjiev, A.S. Džafarov and L.G. Labsker, The asymptotic value of the approximation of functions by a certain family of linear operators, Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk, **3** (1962) 19–28.

[10] H. Karslı, Convergence and rate of convergence by nonlinear singular integral operators depending on two parameters, Appl. Anal., **85** (2006) 781–791.

[11] R.G. Mamedov, On the order of convergence of m-singular integrals at generalized Lebesgue points and in the space $L_p(-\infty,\infty)$, Izv. Akad. Nauk SSSR Ser. Mat., **27**(2) (1963) 287–304.

[12] J. Musielak, On some approximation problems in modular spaces, Constructive Function Theory 1981, (Proc. Int. Conf., Varna, June 1-5, 1981), Publ. House Bulgarian Acad. Sci., Sofia, (1983) 455–461.

[13] J. Musielak, Approximation by nonlinear singular integral operators in generalized Orlicz spaces, Comment. Math., **31** (1991) 79–88.

[14] J. Musielak, Nonlinear approximation in some modular function spaces I., Math. Japonica, **38** (1993) 83–90.

[15] B. Rydzewska, Approximation des fonctions par des intégrales singulières ordinaires, Fasc. Math., **7** (1973) 71–81.

[16] B. Rydzewska, Point-approximation des fonctions par des certaines intégrales singulières, Fasc. Math., **10** (1978) 13–24.

[17] T. Swiderski and E. Wachnicki, Nonlinear singular integrals depending on two parameters, Comment Math., **40** (2000) 181–189.

[18] R. Taberski, Singular integrals depending on two parameters, Prace Mat., **7** (1962) 173-179.

[19] R. Taberski, On double singular integrals, Prace Mat., **19** (1976) 155–160.

[20] G. Uysal and E. Ibikli, A note on nonlinear singular integral operators depending on two parameters, New Trends Math. Sci., **4**(1) (2016) 104–114.

[21] E. Wachnicki and G. Krech, Approximation of functions by nonlinear singular integral operators depending on two parameters, Publ. Math. Debrecen, **92**(3-4) (2018) 481–494.

Dipti Tapiawala[1,2]

[1] Department of Mathematics, C U Shah University, Gujarat, India.

[2] AS and H Department (Mathematics), Sardar Vallabhbhai Patel Institute of Technology, Vasad, Gujarat, India.

E-mail address: tapiawalad@yahoo.com

Gumrah Uysal[3]

[3] Department of Computer Technologies, Division of Technology of Information Security, Karabuk University, Karabuk, Turkey.

E-mail address: fgumrahuysal@gmail.com

Vishnu Narayan Mishra*[4,5]
[4] Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak 484 887, Madhya Pradesh, India.
[5] L. 1627 Awadh Puri Colony Beniganj, Phase - III, Opposite - Industrial Training Institute (I.T.I.), Ayodhya Main Road, Ayodhya 224 001, Uttar Pradesh, India.
E-mail address: vishnunarayanmishra@gmail.com