ON SOME HERMITE-HADAMARD INTEGRAL INEQUALITIES
IN MULTIPLICATIVE CALCULUS

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Abstract. In this paper, we establish some new Hermite-Hadamard integral inequalities for log-φ-convex and φ-convex functions in the framework of multiplicative calculus. Furthermore, some results related to differentiable log-φ-invex functions are also obtained.

1. Introduction

Grossman and Katz [14] initiated the study of Non-Newtonian calculus and modified the classical calculus introduced by Newton and Leibnitz in the 17th century. On the other hands, Bashirov et al. [3] studied the concept of multiplicative calculus and presented a fundamental theorem of multiplicative calculus. Since then a number of interesting results has been obtained in this direction. For more discussion and applications of this discipline, we refer to [28, 2, 3, 4] and [26]. Some elements of stochastic multiplicative calculus have been investigated in [17] and [13]. Bashirov and Riza [5] also studied complex multiplicative calculus.

Another popular Non-Newtonian calculus, known as bigeometric calculus is studied in [29, 15, 1, 18, 27, 6].

Recall that, multiplicative integral called *integral is denoted by \( \int_{a}^{b} f(x)^{dx} \) whereas the ordinary integral is denoted by \( \int_{a}^{b} f(x)dx \). This is due to the fact that the sum of product terms in the definition of a proper Riemann integral of \( f \) on \([a,b]\) is replaced with the product of terms raised to certain powers. It is also known that [3] if \( f \) is positive and Riemann integrable on \([a,b]\), then it is *integrable on \([a,b]\) and

\[
\int_{a}^{b} (f(x))^{dx} = e^{\int_{a}^{b} \ln(f(x))dx}.
\]

Consistent with [3], the following results and notations will be needed in the sequel.

(i) \( \int_{a}^{b} ((f(x))^{p})^{dx} = \int_{a}^{b} ((f(x))^{dx})^{p} \),

(ii) \( \int_{a}^{b} (f(x)g(x))^{dx} = \int_{a}^{b} (f(x))^{dx} \cdot \int_{a}^{b} (g(x))^{dx} \),

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(iii) \( \int_a^b \frac{f(x)}{g(x)} \, dx = \int_a^b \frac{f(x)^{dx}}{g(x)^{dx}} \),

(iv) \( \int_a^b (f(x))^{dx} = \int_a^b (f(x))^{dx} \cdot \int_c^b (f(x))^{dx}, \quad a \leq c \leq b. \)

(v) \( \int_a^b (f(x))^{dx} = 1 \) and \( \int_a^b (f(x))^{dx} = \left( \int_a^b (f(x))^{dx} \right)^{-1} \).

On the other hand, the notion of convexity plays a significant role in many disciplines such as mathematical finance, economics, engineering, management sciences and optimization theory.

In the recent years, several extensions and generalizations of convexity have been investigated. Noor [22] extended the concept of a convex function to \(\phi\)-convex functions. For more results in this direction, we refer to [19] and [22].

Hermite and Hadamard showed independently that the convex functions are related to an integral inequality. Hadamard’s inequality for convex functions has received much attention in recent years and a remarkable variety of refinements and generalizations have been obtained (see for example, [7, 8, 9, 10, 11, 12]).

The aim of this paper is to establish Hermite Hadamard type integral inequalities for log-\(\phi\)-convex functions, and \(\phi\)-convex functions in the setup of multiplicative calculus.

2. Preliminaries

Let \(K\) be a nonempty closed set in \(\mathbb{R}^n\), and \(K^\circ\) the interior of \(K\). We denote by \(\langle \cdot, \cdot \rangle\) and \(\|\cdot\|\) the inner product and norm on \(\mathbb{R}^n\), respectively. Let \(f, \phi : K \to \mathbb{R}\) be continuous mappings.

We recall the following well known results and concepts.

Definition 2.1 A set \(K\) is said to be convex, if for any \(a, b \in K\),

\[
(1 - t)a + tb = a + t(b - a) \in K, \quad \text{for all } t \in [0, 1]. \tag{2.1}
\]

Definition 2.2 A set \(K\) is said to be \(\phi\)-convex, if for any \(a, b \in K\),

\[
a + te^{i\phi}(b - a) \in K, \quad \text{for all } t \in [0, 1]. \tag{2.2}
\]

If we take \(\phi = 0\), then \(\phi\)-convex set becomes a convex set. The converse does not hold in general.

Definition 2.3 The function \(f\) on the convex set \(K\) is said to be convex, if for any \(a, b \in K\), we have

\[
f(a + t(b - a)) = f((1 - t)a + tb) \leq (1 - t)f(a) + tf(b), \quad \text{for all } t \in [0, 1].
\]

The function \(f\) is said to be concave iff \(-f\) is convex.

Definition 2.4 The function \(f\) on the \(\phi\)-convex set \(K\) is said to be \(\phi\)-convex with respect to \(\phi\), if

\[
f(a + te^{i\phi}(b - a)) \leq (1 - t)f(a) + tf(b), \quad \forall a, b \in K, \quad t \in [0, 1].
\]

The function \(f\) is said to be \(\phi\)-concave iff \(-f\) is \(\phi\)-convex. Note that, every convex function is \(\phi\)-convex but the converse does not hold in general.

Definition 2.5 The function \(f\) on the convex set \(K\) is called quasi convex, if

\[
f(a + t(b - a)) \leq \max \{f(a), f(b)\}, \quad \forall a, b \in K, \quad t \in [0, 1].
\]

Definition 2.6 The function \(f\) on the \(\phi\)-convex set \(K\) is called quasi \(\phi\)-convex, if

\[
f(a + te^{i\phi}(b - a)) \leq \max \{f(a), f(b)\}, \quad \forall a, b \in K, \quad t \in [0, 1].
\]
Definition 2.7 The function \( f \) on the convex set \( K \) is called logarithmic convex, if
\[
f(a + t(b - a)) \leq (f(a))^{1-t}(f(b))^t. \tag{2.3}
\]
Moreover, we have
\[
\log f(a + t(b - a)) \leq (1 - t) \log f(a) + t \log f(b) \quad \forall \ a, b \in K, \quad t \in [0, 1].
\]

Definition 2.8 The function \( f \) on the convex set \( K \) is called logarithmic \( \phi \)-convex, if
\[
f(a + te^{i\phi}(b - a)) \leq (f(a))^{1-t}(f(b))^t. \tag{2.4}
\]

Definition 2.9 The function \( f \) on the \( \phi \)-convex set \( K \) is said to be logarithmic \( \phi \)-convex with respect to \( \phi \), if
\[
f(a + te^{i\phi}(b - a)) \leq (f(a))^{1-t}(f(b))^t. \tag{2.4}
\]
Moreover, we have
\[
\log f(a + te^{i\phi}(b - a)) \leq (1 - t) \log f(a) + t \log f(b) \quad \forall \ a, b \in K, \quad t \in [0, 1].
\]

In view of this fact, we have the following.

Definition 2.10 The differentiable function \( f \) on the \( \phi \)-convex set \( K \) is said to be a log-\( \phi \)-invex function with respect to \( \phi \), if
\[
\log f(b) - \log f(a) \geq \left\langle f'_\phi(a), b - a \right\rangle \quad \forall \ a, b \in K.
\]

It is well known [10, 11, 24, 25] that if \( f \) is a convex function on the interval \( I = [a, b] \), then
\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}, \quad \forall \ a, b \in I, \tag{2.5}
\]
which is known as the Hermite-Hadamard inequalities for the convex functions. For some results related to this classical result, we refer to [10, 11, 24, 25] and the references therein.

Dragomir and Mond [10] proved the following Hermite-Hadamard type inequalities for the log-convex functions:
\[
f\left(\frac{a + b}{2}\right) \leq \exp\left[\frac{1}{b - a} \int_a^b \ln[f(x)]dx\right]
\leq \frac{1}{b - a} \int_a^b G(f(x), f(a + b - x))dx
\leq \frac{1}{b - a} \int_a^b f(x)dx
\leq L(f(a), f(b)) \leq \frac{f(a) + f(b)}{2}, \tag{2.6}
\]
where \( G(p, q) = \sqrt{pq} \) is the geometric mean and \( L(p, q) = \frac{p - q}{\ln p - \ln q} \) (for \( p \neq q \)) is the logarithmic mean of the positive real numbers \( p, q \). (For \( p = q \), we put \( L(p, q) = p \).)

From now onward, unless otherwise stated, we assume that $K = [a, a + e^{i\phi} (b - a)]$ and $0 \leq \phi \leq \frac{\pi}{2}$.

Note that, if $K = [a, a + e^{i\phi} (b - a)]$ is an interval, then the $\phi$-convex functions can be characterized as follows:

$$
| \begin{array}{ccc} 1 & 1 & 1 \\ a & x & a + e^{i\phi} (b - a) \\ f(a) & f(x) & f(a + e^{i\phi} (b - a)) \end{array} | \geq 0,
$$

where $x = a + te^{i\phi}(b - a) \in K$.

Using this definition, it can be easily shown that $\phi$-convex functions satisfy the inequalities of the form:

$$
f(x) \leq f(a) + \frac{f(b) - f(a)}{e^{i\phi}(b - a)} (x - a). \quad (2.7)
$$

3. Main Results

**Theorem 3.1.** If $f : K \to (0, \infty)$ is a $\phi$-convex function on the interval of real numbers in $K^\circ$ and $a, b \in K^\circ$ with $a < a + e^{i\phi} (b - a)$ and $0 \leq \phi \leq \frac{\pi}{2}$, then

$$
\left( \int_a^{a + e^{i\phi} (b - a)} (f(x))^\frac{1}{e^{i\phi}(b - a)} \, dx \right) \leq \frac{(f(b))^{f(b)} - f(a)^{f(a)}}{e}. \quad (2.8)
$$

**Proof.** As $f$ is a $\phi$-convex function, we have

$$
\int_a^{a + e^{i\phi} (b - a)} (f(x))^\frac{1}{e^{i\phi}(b - a)} \ln(f(x)) \, dx
$$

$$
= e^{\int_a^{a + e^{i\phi} (b - a)} \ln(f(x)) \, dx} e^{\int_a^{a + e^{i\phi} (b - a)} (f(x))^\frac{1}{e^{i\phi}(b - a)} \, dx}
$$

$$
\leq e^{\int_a^{a + e^{i\phi} (b - a)} (f(x))^\frac{1}{e^{i\phi}(b - a)} \, dx}
$$

$$
= e^{\int_a^{a + e^{i\phi} (b - a)} (\ln(f(x)) - (f(x))^\frac{1}{e^{i\phi}(b - a)} \, dx)
$$

$$
= e^{\int_a^{a + e^{i\phi} (b - a)} (\ln(f(x)) - (f(x))^\frac{1}{e^{i\phi}(b - a)} \, dx)
$$

$$
= e^{\int_a^{a + e^{i\phi} (b - a)} (\ln(f(x)) - (f(x))^\frac{1}{e^{i\phi}(b - a)} \, dx)
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$$
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$$
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$$
= e^{\int_a^{a + e^{i\phi} (b - a)} (\ln(f(x)) - (f(x))^\frac{1}{e^{i\phi}(b - a)} \, dx)
$$

$$
= e^{\int_a^{a + e^{i\phi} (b - a)} (\ln(f(x)) - (f(x))^\frac{1}{e^{i\phi}(b - a)} \, dx)
$$

$$
= e^{\int_a^{a + e^{i\phi} (b - a)} (\ln(f(x)) - (f(x))^\frac{1}{e^{i\phi}(b - a)} \, dx)
$$

Hence

$$
\left( \int_a^{a + e^{i\phi} (b - a)} (f(x))^\frac{1}{e^{i\phi}(b - a)} \, dx \right) \leq \frac{(f(b))^{f(b)} - f(a)^{f(a)}}{e}. \quad (2.8)
$$

\[\square\]
Corollary 3.2. If $f : K = [a, b] \to (0, \infty)$ is a convex function on the interval of real numbers in $K$, and $a, b \in K$, then

$$\left( \int_a^b (f(x))^\varphi dx \right)^\frac{1}{\varphi} \leq \left( \frac{f(b) + f(a)}{2} \right)^\varphi.$$ 

Proof. From Theorem 1 we get this inequality for $\phi = 0$. \hfill \Box

Theorem 3.3. If $f : K \to (0, \infty)$ is a log-$\phi$-convex function on $K$, then

$$\left( \int_a^b (f(x))^\varphi dx \right)^\frac{1}{\varphi} \leq G(f(a), f(b)) \leq L(f(a), f(b)) \leq A(f(a), f(b)),$$

where $G(., .)$, $L(., .)$, $A(., .)$ are geometric, logarithmic and arithmetic means, respectively.

Proof. Since $f$ is a convex function, we have

$$\int_a^{a+e^{\varphi}(b-a)} (f(x))^\varphi dx = e^{-\varphi(b-a)} \int_a^{a+e^{\varphi}(b-a)} (\ln(f(x)))^\varphi dx$$

$$= e^{\varphi(b-a)} \int_0^1 \ln \left( f(a + te^{\varphi}(b-a)) \right) dt$$

$$\leq e^{\varphi(b-a)} \int_0^1 \ln \left( f(a)^{-1} f(b)^{t} \right) dt$$

$$= e^{\varphi(b-a)} \int_0^1 (1-t) \ln f(a) + t \ln f(b) dt$$

$$= e^{\varphi(b-a)} \left\{ \int_0^1 \frac{\ln f(b) - \ln f(a) + t \ln f(b) dt}{2} \right\}$$

$$= e^{\varphi(b-a)} \left\{ \frac{\ln f(b) + \ln f(a)}{2} \right\}$$

$$= (\frac{\ln f(b) + \ln f(a)}{2})^\varphi$$

$$= (f(a), f(b))^\varphi \leq \left( (\frac{f(a) + f(b)}{2})^\varphi \right).$$

Hence,

$$\left( \int_a^b (f(x))^\varphi dx \right)^\frac{1}{\varphi} \leq \sqrt{f(a) \cdot f(b)} = G(f(a), f(b))$$

$$\leq L(f(a), f(b)) \leq \frac{f(a) + f(b)}{2} = A(f(a), f(b)).$$

\hfill \Box
Corollary 3.4. If \( f : K = [a, b] \to (0, \infty) \) is a log convex function on the interval \([a, b]\), then

\[
\left( \int_a^b (f(x))^d \right)^{\frac{1}{d}} = G(f(a), f(b)) \leq L(f(a), f(b)) \leq A(f(a), f(b)).
\]

Proof. From Theorem 3, we obtain this inequality for \( \phi = 0 \).

\[ \square \]

Theorem 3.5. Let \( f, g : K \to (0, \infty) \) be log-\( \phi \)-convex functions on the interval of real numbers in \( K^o \) and \( a, b \in K^o \). Then

\[
\left( \int_a^{a+e^{2\phi}(b-a)} \left( f(x)^{g(x)} \right)^d \right)^{\frac{1}{d}} \leq \sqrt{f(a)f(b).g(a)g(b)} = G(f(a)f(b), g(a)g(b)) \leq L(f(a)f(b), g(a)g(b)) \leq \frac{f(a)f(b) + g(a)g(b)}{2}.
\]

Proof. As \( f, g \) are log-\( \phi \)-convex functions, therefore

\[
\begin{align*}
\int_a^{a+e^{2\phi}(b-a)} \left( f(x)^{g(x)} \right)^d &= e^{\int_a^{a+e^{2\phi}(b-a)} \left( \ln(f(x))g(x) \right)^d dx} \\
&= e^{\int_a^{a} \left( \ln(f(a))g(a) \right)^d dx} e^{\int_a^{e^{2\phi}(b-a)} \left( \ln(f(a))g(a) \right)^d dx} \\
&= e^{\int_a^{a} \left( \ln(f(a))g(a) \right)^d dx} e^{\int_a^{e^{2\phi}(b-a)} \left( \ln(f(a))g(a) \right)^d dx} \\
&= e^{\frac{1}{2} \left( \ln(f(a))g(a) \right)^d dx} e^{\frac{1}{2} \left( \ln(f(a))g(a) \right)^d dx} \\
&= \left( e^{\left( \ln(f(a))g(a) \right)^d dx} \right)^{\frac{1}{2}} e^{\frac{1}{2} \left( \ln(f(a))g(a) \right)^d dx} \\
&= \left( f(a)f(b).g(a)g(b) \right)^{\frac{1}{2}} e^{\frac{1}{2} \left( \ln(f(a))g(a) \right)^d dx} \\
&= \left( G(f(a)f(b), g(a)g(b)) \right)^{\frac{1}{2}} e^{\frac{1}{2} \left( \ln(f(a))g(a) \right)^d dx} \\
&\leq L(f(a)f(b), g(a)g(b))^{\frac{1}{2}} e^{\frac{1}{2} \left( \ln(f(a))g(a) \right)^d dx} \\
&\leq \frac{f(a)f(b) + g(a)g(b)}{2}.
\end{align*}
\]
Hence
\[
\left( \int_a^{a+e^{i\phi}(b-a)} (f(x)g(x))^\frac{1}{2} \right) dx \\
\leq \sqrt{f(a)f(b), g(a)g(b)} \\
= G(f(a)f(b), g(a)g(b)) \\
\leq L(f(a)f(b), g(a)g(b)) \\
\leq A(f(a)f(b), g(a)g(b)).
\]

\[\square\]

Corollary 3.6. If \( f, g : K = [a, b] \to (0, \infty) \) is a log convex functions on the interval of real numbers in \( K^\circ \) and \( a, b \in K^\circ \), then
\[
\left( \int_a^{b} (f(x)g(x))^\frac{1}{2} \right) dx \\
\leq \sqrt{f(a)f(b), g(a)g(b)} = G(f(a)f(b), g(a)g(b)) \\
\leq L(f(a)f(b), g(a)g(b)) \\
\leq A(f(a)f(b), g(a)g(b)).
\]

\textbf{Proof.} This follows from Theorem 5 by taking \( \phi = 0 \). \[\square\]

\textbf{Theorem 3.7.} If \( f, g : K \to (0, \infty) \) are differentiable log-\( \phi \)-invex functions on the interval of real numbers in \( K^\circ \) and \( a, b \in K^\circ \), then
\[
\int_a^{a+e^{i\phi}(b-a)} (2f(x)g(x))^dx \\
\geq \int_a^{a+e^{i\phi}(b-a)} \left[ f \left( \frac{2a+e^{i\phi}(b-a)}{2} \right) g(x) \exp \left[ \left( \frac{f' \left( \frac{2a+e^{i\phi}(b-a)}{2} \right)}{f \left( \frac{2a+e^{i\phi}(b-a)}{2} \right)}, x - \frac{2a+e^{i\phi}(b-a)}{2} \right) \right] \right] dx
\]
\[
\times f(x) \exp \left[ \left( \frac{g' \left( \frac{2a+e^{i\phi}(b-a)}{2} \right)}{g \left( \frac{2a+e^{i\phi}(b-a)}{2} \right)}, x - \frac{2a+e^{i\phi}(b-a)}{2} \right) \right] \right] dx.
\]

\textbf{Proof.} Since \( f, g \) are differentiable log-\( \phi \)-invex functions. So, we have
\[
\log f(x) - \log f(y) \geq \left( \frac{f'(y)}{f(y)}, x - y \right), \text{ and}
\]
\[
\log g(x) - \log g(y) \geq \left( \frac{g'(y)}{g(y)}, x - y \right) \quad \forall \ g(x), g(y) \in K,
\]
which implies that
\[
\log \frac{f(x)}{f(y)} \geq \left( \frac{f'(y)}{f(y)}, x - y \right).
\]

That is,
\[
f(x) \geq f(y) \exp \left[ \left( \frac{f'(y)}{f(y)}, x - y \right) \right] \quad \text{(3.1)}
\]
\[
g(x) \geq g(y) \exp \left[ \left( \frac{g'(y)}{g(y)}, x - y \right) \right]. \quad \text{ (3.2)}
\]
Multiplying on both sides of (3.1) and (3.2) by \(g(x)\) and \(f(x)\), respectively and then adding the resultants, we have

\[
2f(x)g(x) \geq g(x)f(y) + \exp \left[ \frac{f'(y)}{f(y)}(x - y) \right]
\]

\[
	imes \exp \left[ \frac{g'(y)}{g(y)}(x - y) \right] + f(x)g(y)\exp \left[ \frac{g(y)}{f(y)}(x - y) \right].
\]

(3.3)

By taking \(y = \frac{2a + e^{i\phi}(b - a)}{2}\) in (3.3), we obtain that

\[
2f(x)g(x) \geq g(x)f \left( \frac{2a + e^{i\phi}(b - a)}{2} \right) \exp \left[ \frac{f'(x)}{f \left( \frac{2a + e^{i\phi}(b - a)}{2} \right)}(x - 2a + e^{i\phi}(b - a)) \right]
\]

\[
+ f(x)g \left( \frac{2a + e^{i\phi}(b - a)}{2} \right) \exp \left[ \frac{g'(x)}{g \left( \frac{2a + e^{i\phi}(b - a)}{2} \right)}(x - 2a + e^{i\phi}(b - a)) \right],
\]

\[
\int_a^{a + e^{i\phi}(b - a)} (2f(x)g(x)) \, dx
\]

\[
\geq \int_a^{a + e^{i\phi}(b - a)} g(x) \exp \left[ \frac{f'(x)}{f \left( \frac{2a + e^{i\phi}(b - a)}{2} \right)}(x - 2a + e^{i\phi}(b - a)) \right] + g \left( \frac{2a + e^{i\phi}(b - a)}{2} \right) \, dx
\]

\[
\times f(x) \exp \left[ \frac{g'(x)}{g \left( \frac{2a + e^{i\phi}(b - a)}{2} \right)}(x - 2a + e^{i\phi}(b - a)) \right]
\]

\[
\]
Proof. Since \( f, g \) are \( \phi \)-convex functions, we have
\[
\int_a^{a+e^\phi(b-a)} (f(x)g(x))^dx = e^{e^\phi(b-a)} \int_a^{a+e^\phi(b-a)} \ln(f(x)g(x))dx
\]
\[
= e^{e^\phi(b-a)} \int_a^{a+e^\phi(b-a)} \ln((f(a)+tf(b))(1-t))^2 \, dt
\]
\[
= e^{e^\phi(b-a)} \int_a^{a+e^\phi(b-a)} \ln(f(a)+tf(b)-f(a)) \, dt
\]
\[
= e^{e^\phi(b-a)} \ln(f(a)+tf(b)-f(a)) \int_a^{a+e^\phi(b-a)} \frac{1}{f(a)+tf(b)-f(a)} \, dt
\]
\[
= e^{e^\phi(b-a)} \ln(f(b)(b)-f(a)(a)) \int_a^{a+e^\phi(b-a)} \frac{1}{f(b)(b)-f(a)(a)} \, dt
\]
\[
= e^{e^\phi(b-a)} \ln(f(b)(b)-f(a)(a)) \int_a^{a+e^\phi(b-a)} \frac{1}{f(b)(b)-f(a)(a)} \, dt
\]
\[
= e^{e^\phi(b-a)} \ln(f(b)(b)-f(a)(a)) \int_a^{a+e^\phi(b-a)} \frac{1}{f(b)(b)-f(a)(a)} \, dt
\]
\[
\int_a^{a+e^\phi(b-a)} (f(x)g(x))^dx \leq e^{e^\phi(b-a)} \ln(f(b)(b)-f(a)(a)) \int_a^{a+e^\phi(b-a)} \frac{1}{f(b)(b)-f(a)(a)} \, dt
\]

Hence
\[
\left( \int_a^{a+e^\phi(b-a)} (f(x)g(x))^dx \right)^{\frac{1}{e^{e^\phi(b-a)}}} \leq e^{e^\phi(b-a)} \ln(f(b)(b)-f(a)(a)) \int_a^{a+e^\phi(b-a)} \frac{1}{f(b)(b)-f(a)(a)} \, dt
\]

Corollary 3.10. If \( f, g : K = [a, b] \to (0, \infty) \) are convex functions on the interval of real numbers in \( K^\circ \) (the interior of \( K \)) and \( a, b \in K^\circ \), then
\[
\left( \int_a^b (f(x)g(x))^dx \right)^{\frac{1}{e^{e^\phi(b-a)}}} \leq e^{e^\phi(b-a)} \ln(f(b)(b)-f(a)(a)) \int_a^b \frac{1}{f(b)(b)-f(a)(a)} \, dx
\]

Proof. Take \( \phi = 0 \) in Theorem 9. 

Theorem 3.11. If \( f, g : K \to (0, \infty) \) are \( \phi \)-convex and log-\( \phi \)-convex functions, respectively on the interval of real numbers \( K^\circ \) and \( a, b \in K^\circ \), then
\[
\left( \int_a^{a+e^\phi(b-a)} (f(x)g(x))^dx \right)^{\frac{1}{e^{e^\phi(b-a)}}} \leq e^{e^\phi(b-a)} \ln(f(b)(b)-f(a)(a)) \int_a^{a+e^\phi(b-a)} \frac{1}{f(b)(b)-f(a)(a)} \, dt
\]
\[
\leq e^{e^\phi(b-a)} \ln(f(b)(b)-f(a)(a)) \int_a^{a+e^\phi(b-a)} \frac{1}{f(b)(b)-f(a)(a)} \, dt
\]
\[
\leq e^{e^\phi(b-a)} \ln(f(b)(b)-f(a)(a)) \int_a^{a+e^\phi(b-a)} \frac{1}{f(b)(b)-f(a)(a)} \, dt
\]
Proof. Let \( f, g \) be \( \phi \)-convex and log-\( \phi \)-convex functions, respectively. Then

\[
\int_a^{a+e^{i\phi}(b-a)} (f(x)g(x)) \, dx = e^{\int_a^{a+e^{i\phi}(b-a)} \ln(f(x)g(x)) \, dx}
\]

\[
= e^{e^{i\phi}(b-a)} \int_0^1 \ln(f(a+tf-a)g(a+tf-a)) \, dt
\]

\[
\leq e^{e^{i\phi}(b-a)} \int_0^1 (\ln(1-t) f(a)+tf(b)) \ln((g(a))^{1-t}(g(b))^t) \, dt
\]

\[
= e^{e^{i\phi}(b-a)} \int_0^1 (\ln(1-t) f(a)+tf(b)) (1-t) \ln(g(a)) + t \ln(g(b)) \, dt
\]

\[
= e^{e^{i\phi}(b-a)} \{ \int_0^1 (\ln(1-t) f(a)+tf(b)) \, dt + e^{\int_0^1 (1-t) \ln(g(a)) + t \ln(g(b))) \, dt \}
\]

\[
= e^{e^{i\phi}(b-a)} \{ \ln f(b) - (f(b) - f(a)) \int_0^1 \frac{1}{f(b) - f(a)} \, dt + \ln(g(a), g(b)) \}
\]

\[
= e^{e^{i\phi}(b-a)} \{ \ln f(b) - 1 + \ln(G(g(a), g(b))) \}
\]

\[
= \left[ \frac{(f(b))^{\frac{f(a)}{f(b)-f(a)}} \cdot (f(a))^{\frac{f(a)}{f(b)-f(a)}} \cdot G(g(a), g(b))}{e} \right] e^{e^{i\phi}(b-a)}
\]

Hence

\[
\left( \int_a^{a+e^{i\phi}(b-a)} (f(x)g(x)) \, dx \right)^{\frac{1}{e^{i\phi}(b-a)}}
\]

\[
\leq \left( \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}} \cdot (f(a))^{\frac{f(a)}{f(b)-f(a)}} \cdot G(g(a), g(b))}{e} \right)
\]

\[
\leq \left( \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}} \cdot (f(a))^{\frac{f(a)}{f(b)-f(a)}} \cdot L(g(a), g(b))}{e} \right)
\]

\[
\leq \left( \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}} \cdot (f(a))^{\frac{f(a)}{f(b)-f(a)}} \cdot A(g(a), g(b))}{e} \right)
\]
Corollary 3.12. Let $f, g : K = [a, b] \rightarrow (0, \infty)$ are convex and log convex functions, respectively on the interval of real numbers in $K^\circ$ and $a, b \in K^\circ$, then

\[
\left( \int_a^b (f(x)g(x)) \, dx \right)^{\frac{1}{b-a}} \leq \left( \frac{f(b)}{\tau_{a,b}^{f(b)}} \cdot \frac{f(a)}{\tau_{a,b}^{f(a)}} \cdot \frac{G(g(a), g(b))}{e} \right) \leq \left( \frac{f(b)}{\tau_{a,b}^{f(b)}} \cdot \frac{f(a)}{\tau_{a,b}^{f(a)}} \cdot \frac{L(g(a), g(b))}{e} \right) \leq \left( \frac{f(b)}{\tau_{a,b}^{f(b)}} \cdot \frac{f(a)}{\tau_{a,b}^{f(a)}} \cdot \frac{A(g(a), g(b))}{e} \right).
\]

Proof. The result follows from Theorem 11, if we take $\phi = 0$. \qed

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References


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