

TWO-SIDED INEQUALITIES FOR THE LEMNISCATE FUNCTIONS

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ABSTRACT. Inequalities involving lemniscate functions are obtained. Derived inequalities possess the same structure as the generalized Wilker and Huygens inequalities for trigonometric, hyperbolic and Jacobian elliptic functions and for the Schwab-Borchardt means as well. Mitrinović-Adamović and Cusa-Huygens type inequalities for the lemniscate sine function are also established.

1. INTRODUCTION

The arc length of Bernoulli's lemniscate $r^2 = \cos 2\theta$ from the origin to the point with radial coordinate x is $\operatorname{arcsl}(x)$, where arcsl is the arc lemniscate sine function studied by C.F. Gauss in 1797–1798. Another lemniscate function investigated by Gauss is the hyperbolic arc lemniscate sine. Both functions appear in the formula for the lemniscatic mean and their definitions are given in the next section.

In this paper, which is a continuation of investigations reported in [5] and [7], we present two-sided inequalities involving lemniscate functions and the lemniscatic mean. Notation and definitions are given in Section 2. In the next section we include two lemmas used in the proofs of the two inequalities in Section 4. The Mitrinović-Adamović and the Cusa-Huygens type inequalities for the lemniscate sine function are established in Section 5.

2. NOTATION AND DEFINITIONS

We begin this section with definitions of four lemniscate functions which will be used in subsequent sections of this paper.

Gauss' arc lemniscate sine and the hyperbolic arc lemniscate sine are defined, respectively, as

$$\operatorname{arcsl} x = \int_0^x \frac{dt}{\sqrt{1-t^4}} \quad (2.1)$$

($|x| \leq 1$) and

$$\operatorname{arcslh} x = \int_0^x \frac{dt}{\sqrt{1+t^4}} \quad (2.2)$$

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($x \in \mathbb{R}$). See [1, p. 259], [2, (2.5)–(2.6)], [12, Ch. 1]). We will also utilize another pair of lemniscate functions, the arc lemniscate tangent arctl and the hyperbolic arc lemniscate tangent arctlh . Both functions have been introduced in [5, (3.1)–(3.2)]. Therein it has been proven that

$$\text{arctl } x = \text{arcsl} \left(\frac{x}{\sqrt[4]{1+x^4}} \right), \quad x \in \mathbb{R} \quad (2.3)$$

and

$$\text{arctlh } x = \text{arcslh} \left(\frac{x}{\sqrt[4]{1-x^4}} \right), \quad |x| < 1 \quad (2.4)$$

(see [5, Prop. 3.1]). It is worth mentioning that all four lemniscate functions can be expressed in terms of the completely symmetric elliptic integral of the first kind

$$R_F(x, y, z) = \frac{1}{2} \int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2} dt,$$

where at most one of the nonnegative variables x, y, z is 0 (see [3, (9.2-1)]).

The lemniscatic mean $LM(x, y) \equiv LM$ of $x > 0$ and $y \geq 0$ is the iterative mean, i.e.,

$$LM(x, y) = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n,$$

where

$$x_0 = x, \quad y_0 = y, \quad x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = (x_{n+1}x_n)^{1/2} \quad (2.5)$$

($n = 0, 1, \dots$). It is known that

$$LM(x, y) = \begin{cases} \frac{\sqrt{x^2 - y^2}}{(\text{arcsl } \sqrt[4]{1 - (y/x)^2})^2}, & y < x \\ \frac{\sqrt{y^2 - x^2}}{(\text{arcslh } \sqrt[4]{(y/x)^2 - 1})^2}, & x < y \\ x, & x = y \end{cases} \quad (2.6)$$

(see [2, (2.7)], [1, (8.5.7)–(8.5.8)]).

For the reader's convenience, let us record some elementary properties of this mean.

- (i) $LM(x, y)$ increases with an increase in either x or y .
- (ii) $LM(x, y) \neq LM(y, x)$.
- (iii) $LM(\lambda x, \lambda y) = \lambda LM(x, y)$ for $\lambda > 0$.

3. LEMMAS

In what follows we will assume that u and v are positive and unequal numbers. The following lemmas will be utilized in the next section of this paper.

Lemma 3.1 ([10]). *If $uv > 1$, then*

$$\frac{1}{u} + \frac{1}{v} < u + v.$$

Lemma 3.2 ([9]). *Let $\alpha > 0$, $\beta > 0$ with $\alpha + \beta = 1$. If*

$$1 < \alpha \frac{1}{u} + \beta \frac{1}{v} < \alpha u + \beta v, \quad (3.1)$$

then

$$1 < \alpha \frac{1}{u^p} + \beta \frac{1}{v^p} < \alpha u^p + \beta v^p. \quad (3.2)$$

The first inequality in (3.2) is valid for $p \geq 1$ while the second one holds true provided $p > 0$.

4. WILKER AND HUYGENS TYPE INEQUALITIES FOR THE LEMNISCATE FUNCTIONS

The main results of this section are the inequalities (4.1) and (4.4). They bear a strong resemblance of the Wilker-type and Huygens-type inequalities for trigonometric functions, hyperbolic functions, Jacobian elliptic functions and also for the Schwab-Borchardt mean. See [15], [16], [10], [6], [8]. In what follows we will assume that x and y are positive and unequal numbers and we will write LM for $LM(x, y)$.

We begin with the following.

Theorem 4.1. *We have*

$$2 < \left(\frac{x}{LM}\right)^{3p} + \left(\frac{y}{LM}\right)^{2p} < \left(\frac{LM}{x}\right)^{3p} + \left(\frac{LM}{y}\right)^{2p}, \quad (4.1)$$

where the first inequality holds true for $p \geq 1$ while the second one is valid for all $p > 0$.

Proof. We shall establish first the left inequality in (4.1) when $p = 1$, i.e.,

$$\left(\frac{x}{LM}\right)^3 + \left(\frac{y}{LM}\right)^2 > 2. \quad (4.2)$$

To this aim we utilize the second inequality in

$$(x^3 y^2)^{1/5} < LM < \frac{3x + 2y}{5} \quad (4.3)$$

(see [5, (4.4)]) to obtain

$$\frac{y}{LM} > \frac{1}{2} \left(5 - 3 \frac{x}{LM}\right) = \frac{1}{2}(5 - 3a),$$

where $a = \frac{x}{LM}$. This implies the inequality

$$\left(\frac{x}{LM}\right)^3 + \left(\frac{y}{LM}\right)^2 > a^3 + \frac{1}{4}(5 - 3a)^2.$$

In order to establish (4.2) it suffices to show that

$$a^3 + \frac{1}{4}(5 - 3a)^2 > 2$$

or what is the same that

$$4a^3 + (5 - 3a)^2 - 8 > 0.$$

One can easily verify that the last inequality can be written as

$$(a - 1)^2(4a + 17) > 0$$

which is satisfied because $a \neq 1$ and $a > 0$.

For the proof of the second inequality in (4.1) when $p = 1$ we write the left inequality in (4.3) as

$$1 < \left(\frac{LM}{x}\right)^3 \left(\frac{LM}{y}\right)^2$$

and next use Lemma 3.1 with $u = \left(\frac{LM}{x}\right)^3$ and $v = \left(\frac{LM}{x}\right)^2$ together with (4.2) to obtain

$$2 < \left(\frac{x}{LM}\right)^3 + \left(\frac{y}{LM}\right)^2 < \left(\frac{LM}{x}\right)^3 + \left(\frac{LM}{y}\right)^2.$$

Dividing each side by 2 and next using Lemma 3.2 with $\alpha = \beta = 1/2$ and u and v as defined above, we obtain the assertion. The proof is complete. \square

Our next result reads as follows.

Theorem 4.2. *If $p \geq 1$, then*

$$5 < 3\left(\frac{x}{LM}\right)^p + 2\left(\frac{y}{LM}\right)^p < 3\left(\frac{LM}{x}\right)^p + 2\left(\frac{LM}{y}\right)^p, \quad (4.4)$$

where the second inequality in (4.4) is valid for $p > 0$.

Proof. We shall prove first the inequality (4.4) when $p = 1$, i.e.,

$$5 < 3\frac{x}{LM} + 2\frac{y}{LM} < 3\frac{LM}{x} + 2\frac{LM}{y}. \quad (4.5)$$

The first inequality in (4.5) follows immediately from the second inequality in (4.3). In the proof of the second inequality in (4.5) we will use quantities a and c , where $a = \frac{LM}{x}$ and $c = \frac{y}{x}$. Then $\frac{LM}{y} = \frac{LM}{x} \cdot \frac{x}{y} = \frac{a}{c}$. Thus the inequality in question can be written as

$$3\frac{1}{a} + 2\frac{c}{a} < 3a + 2\frac{a}{c}$$

or what is the same that

$$a^2 > \frac{c(2c+3)}{3c+2}. \quad (4.6)$$

Using the inequality $LM > (xA^4)^{1/5}$ (see [5, (4.4)]) we have

$$\left(\frac{LM}{x}\right)^5 > \left(\frac{A}{x}\right)^4 = \left(\frac{1+c}{2}\right)^4.$$

Thus $a^5 > \left(\frac{1+c}{4}\right)^4$ or what is the same that

$$a^2 > \left(\frac{1+c}{2}\right)^{8/5}. \quad (4.7)$$

We shall prove (4.6) showing that

$$\left(\frac{1+c}{2}\right)^{8/5} > \frac{c(2c+3)}{3c+2}. \quad (4.8)$$

To prove the inequality (4.8) we let $t = \left(\frac{1+c}{2}\right)^{1/5}$. Hence $c = 2t^5 - 1$ and (4.8) can be written in terms of t as

$$t^8 > \frac{(2t^5 - 1)(4t^5 + 1)}{6t^5 - 1}.$$

Let

$$\begin{aligned} f(t) &:= t^8(6t^5 - 1) - (2t^5 - 1)(4t^5 + 1) \\ &= 6t^{13} - 8t^{10} - t^8 + 2t^5 + 1. \end{aligned}$$

In order to prove the last inequality it suffices to show that $f(t) > 0$ for $t > 0$ and $t \neq 1$. Since $f(1) = f'(1) = 0$,

$$f(t) = (t-1)^2(6t^{11} + 12t^{10} + 8t^9 + 16t^8 + 14t^7 + 11t^6 + 8t^5 + 5t^4 + 4t^3 + 3t^2 + 2t + 1).$$

Clearly $f(t) > 0$. This completes the proof of (4.5). To prove the inequality (4.4) we divide each member of (4.5) by 5 and next apply Lemma 3.2 with $\alpha = \frac{3}{5}$, $\beta = \frac{2}{5}$, $u = \frac{LM}{x}$, $v = \frac{LM}{y}$. The proof is complete. \square

Inequalities (4.1) and (4.4) and the following formulas [7, (4.2)–(4.5)]

$$LM(1, \sqrt{1-x^4}) = \left(\frac{x}{\operatorname{arcsl} x}\right)^2, \quad LM(\sqrt{1-x^4}, 1) = \left(\frac{x}{\operatorname{arctlh} x}\right)^2 \quad (4.9)$$

($|x| < 1$ and

$$LM(1, \sqrt{1+x^4}) = \left(\frac{x}{\operatorname{arcslh} x}\right)^2, \quad LM(\sqrt{1+x^4}, 1) = \left(\frac{x}{\operatorname{arctl} x}\right)^2 \quad (4.10)$$

($x \in \mathbb{R}$) can be used to obtain four generalized Wilker-type inequalities and four generalized Huygens-type inequalities for the lemniscate functions. For instance, it follows from (4.1) and the first formula in (4.9) that

$$\begin{aligned} 2 &< \left(\frac{\operatorname{arcsl} x}{x}\right)^{6p} + (1-x^2)^p \left(\frac{\operatorname{arcsl} x}{x}\right)^{4p} \\ &< \left(\frac{x}{\operatorname{arcsl} x}\right)^{6p} + (1-x^2)^{-p} \left(\frac{x}{\operatorname{arcsl} x}\right)^{4p} \end{aligned}$$

with the domain of validity in p as stated in Theorem 4.1. The remaining seven inequalities can be obtained in the same way. We omit further details.

5. MITRINOVIĆ-ADAMOVIĆ AND CUSA-HUYGENS TYPE INEQUALITIES FOR THE LEMNISCATE FUNCTIONS

A two-sided trigonometric inequality

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2 + \cos x}{3} \quad (5.1)$$

($0 < |x| \leq \frac{\pi}{2}$), where the left inequality was discovered by Mitrinović and Adamović (see, e.g., [4, p. 238]) while the right one is attributed to Cusa and Huygens (see [11]) has attracted attention of several researchers. Similar inequalities for the hyperbolic functions and the Jacobian elliptic functions have been obtained recently. For more details the interested reader is referred to [10], [15], [16], [6] and the references therein.

The goal of this section is to obtain an inequality, similar to (5.1), which involves the lemniscate functions sl and cl . The former is the inverse function of arcsl (see (2.1)) while the latter is the inverse function of the arc lemniscate cosine arcscl :

$$\operatorname{arcscl} x = \int_x^1 \frac{dt}{\sqrt{1-t^4}}$$

(see [14]). It is known that [14, p. 524]

$$slx = k \operatorname{sd}(x/k, k) \quad \text{and} \quad clx = cn(x/k, k),$$

where $k = 1/\sqrt{2}$, sd and cn are the Jacobian elliptic functions with modulus k .

Recall the the number

$$\omega := \operatorname{arcsl}(1) = \frac{(\Gamma(\frac{1}{4}))^2}{4\sqrt{2\pi}} = 1.311\dots$$

is called the first lemniscate constant and is equal to the quarter of the period of the function sl .

The main result of this section reads as follows.

Theorem 5.1. *Let $0 < |x| \leq \omega$ and let $p(x) = 2(clx)/(1 + cl^2x)$. Then*

$$p(x)^{1/5} < \frac{slx}{x} < \left(\frac{3 + 2p(x)}{5}\right)^{1/2}. \quad (5.2)$$

Proof. We let $x := slx$ in the first formula in (4.9) to obtain

$$LM(1, (1 - sl^4x)^{1/2}) = \left(\frac{slx}{x}\right)^2. \quad (5.3)$$

Application of the left inequality in (4.3) to the left side of (5.3) gives

$$(1 - sl^4x)^{1/5} < \left(\frac{slx}{x}\right)^2. \quad (5.4)$$

Making use of

$$sl^2x = \frac{1 - cl^2x}{1 + cl^2x}$$

(see [14, p. 525], [13, p. 46]) we obtain

$$1 - sl^4x = \left(\frac{2clx}{1 + cl^2x}\right)^2 = p^2(x). \quad (5.5)$$

This in conjunction with (5.4) gives the left inequality in (5.2). In order to complete the proof of (5.2) we apply (5.5) to the left side of (5.3) to obtain

$$LM(1, p(x)) = \left(\frac{slx}{x}\right)^2. \quad (5.6)$$

Application of the right inequality in (4.3) gives

$$LM((1, p(x)) < \frac{3 + 2p(x)}{5}.$$

This in conjunction with (5.6) yields the assertion. The proof is complete. \square

REFERENCES

- [1] J.M. Borwein, P.B. Borwein, *Pi and AGM: A Study in Analytic Number Theory and Computational Complexity*, John Wiley and Sons, New York, 1987.
- [2] B.C. Carlson, *Algorithms involving arithmetic and geometric means*, Amer. Math. Monthly, **78**, (1971) 496–505.
- [3] B.C. Carlson, *Special Functions of Applied Mathematics*, Academic Press, New York, 1977.
- [4] D.S. Mitrinović, *Analytic Inequalities* Springer-Verlag, Berlin, 1970.
- [5] E. Neuman, *On Gauss lemniscate functions and lemniscatic mean*, Math. Pannon, **18** (2007), 77–94.
- [6] E. Neuman, *One- and two-sided inequalities for Jacobian elliptic functions and related results*, Integral Transforms Spec. Functions **21** (2010), 399–407.
- [7] E. Neuman, *Inequalities and bounds for the lemniscate functions*, submitted.
- [8] E. Neuman, *Inequalities for the Schwab-Borchardt mean and their applications*, submitted.
- [9] E. Neuman, *On Wilker and Huygens type inequalities*, submitted.

- [10] E. Neuman, J. Sándor, *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker and Huygens inequalities*, Math. Inequal. Appl., **13** (2010), 715–723.
- [11] J. Sándor, M. Bencze, *On Huygens' trigonometric inequality*, RGMIA Res. Rep. Collection, **8** (2005), Issue 3, Article 14.
- [12] C.L. Siegel, *Topics in Complex Function Theory*, Vol. 1, Wiley, New York, 1969.
- [13] K. Ueno, K. Shiga, S. Morita, *A Mathematical Gift II*, Amer. Math. Soc., Providence, Rhode Island, 2004.
- [14] E.T. Whittaker, G.N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge Univ. Press, New York, 1990.
- [15] S. Wu, A. Baricz, *Generalizations of Mitrinović. Adamović and Lazarević's inequalities and their applications*, Publ. Math. Debrecen **75/3-4** (2009), 447–458.
- [16] L. Zhu, *Inequalities for hyperbolic functions and their applications*, J. Inequal. Appl., Volume 2010, Article ID 130821.

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