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ON COMPOSITION OF FRACTIONAL Q-INTEGRAL OPERATORS INVOLVING BASIC HYPERGEOMETRIC FUNCTION

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Abstract. In this paper a new fractional $q-$integral operator using the basic hypergeometric function ${}_{r+1}\phi_r$ as kernel is defined, and a composition formula for a particular type of these operators is established. Further, some particular cases are obtained.

1. Introduction

The concept of differ-integral of complex order ν, which is a generalization of the ordinary $n-th$ derivative and n times integral to any complex number, can be introduced in several ways. The most widely used definition of an integral of fractional order is via an integral transform, called the Riemann-Liouville operator of fractional integration: [13, p. 146]

$$\begin{aligned} {}_aI_x^\alpha\varphi(x) = & \tfrac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)\ dt \quad \mathrm{Re}(\alpha)>0, \\ & = \tfrac{d^n}{dx^n}\ {}_aI_x^{\alpha+n}\varphi(x) \qquad -n<\mathrm{Re}(\alpha)\leq 0,\ n\in\mathbb{N}. \end{aligned} \tag{1.1}$$

Many authors ([1]-[3], [6]-[11], [14]-[17]) have defined and studied operators of fractional integration through an integral transform. Some of these operators are:

1.1. Erdélyi-Kober Operator: [6, p. 4, No. (20)].

$$\begin{aligned} I_{\eta,\alpha}f(x) &= \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^\eta f(t)dt,\ \mathrm{Re}(\alpha)\geq 0, \\ &= x^{-\alpha-\eta}\frac{d^n}{dx^n}x^{\eta+\alpha+n}I_{\eta,\alpha+n}f(x),\ \ \mathrm{Re}(\alpha)<0, \end{aligned} \tag{1.2}$$

where n is the minor integer major that α.

1.2. Saxena operator: [17, p. 288, No. (1)].

$$\mathfrak{T}\left[f(x)\right] = \mathfrak{T}\left[\alpha,\beta,\gamma,m;f(x)\right]$$

$$= \frac{x^{-\gamma-1}}{\Gamma(1-\alpha)}\int_0^x F(\alpha,\beta+m;\beta;\frac{t}{x})t^\gamma f(t)dt, \quad \mathrm{Re}(\alpha)<1;\left|\frac{t}{x}\right|<1, \tag{1.3}$$

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where $F(a,b;c;x)$ denotes the Gauss hypergeometric function, and the parameters involved are complex numbers.

1.3. **Kalla operators: [7, p. 93, No. (2.1)].**

$$R^{m,n,p,q;a}_{\gamma,r;a_i,b_i}[f(x)] = \frac{x^{-\gamma-1}\prod_{i=1}^{q}\Gamma(1-b_i)\,r}{\prod_{i=1}^{m}\Gamma(a_i)\prod_{i=1}^{p}\Gamma(1-a_i)}\int_0^x G^{m,n}_{p,q}\left[a\left(\frac{t}{x}\right)^r \middle| \begin{array}{c} a_i \\ b_i \end{array}\right] t^{\gamma} f(t)\, dt, \tag{1.4}$$

where r is positive integer, m,n,p,q are integers such that $0 \le m \le q, 0 \le n \le p, 2(m+n) > p+q, |\arg a| < \left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi, \gamma, a_i$ and b_i are complex parameters.

On using the following relation [12, p. 727]

$$ {}_pF_q\left[\begin{array}{c} a_1,...,a_p \\ b_1,...,b_q \end{array}; -x\right] = \Gamma\left[\begin{array}{c} b_1,b_2,...,b_q \\ a_1,a_2,...,a_p \end{array}\right] \times$$

$$G^{1,p}_{p,q+1}\left[x \middle| \begin{array}{c} 1-a_1,...,1-a_p \\ 0,1-b_1,...,1-b_q \end{array}\right],$$

operator (1.4) leads to

$$R^{1,p,p,q+1;a}_{\gamma,1;1-a_1,...,1-a_p,0,1-b_1,...,1-b_q}[f(x)] = \frac{x^{-\gamma-1}}{\Gamma(1-a_1)} \times$$

$$\int_0^x {}_pF_q\left[\begin{array}{c} a_1,...,a_p \\ b_1,...,b_q \end{array}; -a\left(\frac{t}{x}\right)\right] t^{\gamma} f(t)\, dt, \tag{1.5}$$

if one of the following conditions holds [12, p. 437, No. 7.2.3.1]

1) $p \le q$. 2) $p = q+1,\ \left|-a\left(\frac{t}{x}\right)\right| < 1$.
3) $p = q+1,\ \left|-a\left(\frac{t}{x}\right)\right| = 1,\ \operatorname{Re}\mu > 0$.
4) $p = q+1,\ \left|-a\left(\frac{t}{x}\right)\right| = 1,\ -a\left(\frac{t}{x}\right) \ne 1,\ -1 < \operatorname{Re}\mu \le 0$,

where $\mu = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j,\ \gamma,\ a_i$ and b_i are complex parameters.

1.4. **Basic analogue of Kober fractional $q-$integral operator.** A basic analogue of Kober fractional $q-$integral operator has been defined by Agarwal [1] in the following form:

$$\mathrm{I}^{\eta,\mu}_q f(x) = \frac{x^{-\eta-1}}{\Gamma_q(\mu)}\int_0^x t^{\eta}\left(\frac{tq}{x};q\right)_{\mu-1} f(t)\, d_qt, \quad \eta,\mu \in \mathbb{C},\ Re(\mu) > 0, \tag{1.6}$$

where μ is an arbitrary order of integration, and

$$(a;q)_{\alpha} = \frac{(a;q)_{\infty}}{(aq^{\alpha};q)_{\infty}}, \quad \alpha \in \mathbb{R}. \tag{1.7}$$

1.5. **The operator** $L(.)$**.** We consider the operator $L(.)$ introduced by Delgado and Galué [2, p. 63, No. (39)] in the following form:

$$L\{l, b_1, b_2, ..., b_r, \gamma, m_1, m_2, ..., m_r; f(x)\} =$$

$$\frac{x^{-\gamma-1}}{\Gamma_q(l+1)} \int_0^x t^{\gamma}\, {}_{r+1}\phi_r \left[\begin{array}{l} q^{-l}, q^{m_1+b_1}, ..., q^{m_r+b_r} \\ q^{b_1}, ..., q^{b_r} \end{array} ; q, q\frac{t}{x} \right] f(t)\ d_qt, \tag{1.8}$$

$$l, m_1, ..., m_r \text{ nonnegative integers},\ \gamma \in \mathbb{C},\ b_1, ..., b_r \neq 0, -1, -2, ..., \left|\frac{t}{x}\right| < 1.$$

In this paper a new fractional $q-$integral operator using the basic hypergeometric function ${}_{r+1}\phi_r$ as kernel is defined, and a composition formula for a particular type of these operators is established. Further, some particular cases are obtained.

2. Basic Hypergeometric Series

In this section we present some definitions necessary for the development of the next sections.

2.1. **The $q-$shifted factorial.** The $q-$shifted factorial is defined as: [4, 5]

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}), & n = 1, 2... \\ [(1-aq^{-1})(1-aq^{-2})...(1-aq^{-n})]^{-1}, & n = -1, -2, ... \end{cases} \tag{2.1}$$

Also,

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n = \prod_{k=0}^{\infty} (1-aq^k), \tag{2.2}$$

which converges for $|q| < 1$ and diverges for $a \neq 0$ and $|q| \geqslant 1$, and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad n \in Z, |q| < 1. \tag{2.3}$$

2.2. **Identities.** We recall here the following q-identities given by Gasper and Rahman [5, p. 6, Nos. (1.2.33), (1.2.35), (1.2.38); p. 20, No. 1.1(i)]:

$$(a;q)_{n+k} = (a;q)_n (aq^n;q)_k. \tag{2.4}$$

$$(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}. \tag{2.5}$$

$$(aq^{-n};q)_k = \frac{(a;q)_k (qa^{-1};q)_n}{(a^{-1}q^{1-k};q)_n} q^{-nk}. \tag{2.6}$$

$$(aq^{-n};q)_n = (q/a;q)_n \left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}}. \tag{2.7}$$

2.3. **The $q-$integral.** Thomae (1869) and Jackson (1910) introduced the q-integral by: [5, p. 19, Nos. (1.11.2), (1.11.3)]

$$\int_a^b f(t) d_qt = \int_0^b f(t) d_qt - \int_0^a f(t) d_qt,$$

where

$$\int_0^a f(t) d_qt = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n. \tag{2.8}$$

2.4. **Basic hypergeometric series.** This series was introduced by Heine (1846), it is defined as: [4, 5]

$$ {}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n, \tag{2.9} $$

where it is assumed that $c \neq q^{-m}$ for $m = 0, 1, ...,$ and $(a;q)_n$ is the $q-$shifted factorial defined in (2.1).

2.5. **Generalized basic hypergeometric series.** A generalization of the basic hypergeometric series ${}_2\phi_1$, is given by: [4, 5]

$$ {}_r\phi_s(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; q, z) \equiv {}_r\phi_s \left[\begin{matrix} a_1, a_2, ..., a_r \\ b_1, b_2, ..., b_s \end{matrix} ; q, z \right] = $$

$$ \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n ... (a_r;q)_n}{(q;q)_n (b_1;q)_n ... (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n, \tag{2.10} $$

where $b_1, ..., b_s \neq q^{-m}$ for $m = 0, 1, ...; \binom{n}{2} = \frac{n(n-1)}{2}$; $q \neq 0$ when $r > s+1$ and $\lim\limits_{q \to 1^-} {}_r\phi_s = {}_rF_s$.

By reversing the order of summation [5, p. 21, No. 1.4(ii)]

$$ {}_{r+1}\phi_s \left[\begin{matrix} a_1, a_2, ..., a_r, q^{-n} \\ b_1, b_2, ..., b_s \end{matrix} ; q, z \right] = \frac{(a_1, ..., a_r; q)_n}{(b_1, ..., b_s; q)_n} \left(\frac{z}{q} \right)^n \left((-1)^n q^{\binom{n}{2}} \right)^{s-r-1} \times $$

$$ \sum_{k=0}^{n} \frac{(q^{1-n}/b_1, ..., q^{1-n}/b_s, q^{-n}; q)_k}{(q, q^{1-n}/a_1, ..., q^{1-n}/a_r; q)_k} \left(\frac{b_1 ... b_s}{a_1 ... a_r} \frac{q^{n+1}}{z} \right)^k, \tag{2.11} $$

with $n = 0, 1, ...$

2.6. **$q-$analogue of the Karlsson-Minton summation formula.** Gasper (1981) derived a $q-$analogue of the Karlsson-Minton summation formula, which is given by: [5, p. 16, No. (1.9.10)]

$$ {}_{r+2}\phi_{r+1} \left[\begin{matrix} q^{-n}, b, b_1 q^{m_1}, ..., b_r q^{m_r} \\ bq, b_1, ..., b_r \end{matrix} ; q, q \right] = $$

$$ \frac{b^n (q;q)_n}{(bq;q)_n} \frac{(b_1/b;q)_{m_1} ... (b_r/b;q)_{m_r}}{(b_1;q)_{m_1} ... (b_r;q)_{m_r}}, \ n \geq m_1 + ... + m_r \tag{2.12} $$

$m_1, m_2, ..., m_r$ nonnegative integers.

2.7. **The $q-$binomial theorem.** One of the most important summation formulas for hypergeometric series is given by the binomial theorem:

$$ {}_2F_1(a,c;c;z) = {}_1F_0(a;-;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}, \quad |z| < 1, \tag{2.13} $$

whose $q-$analogue, derived by Cauchy (1843), Heine (1847) and other mathematicians, is [4, 5]

$$ {}_1\phi_0(a;-;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1, |q| < 1. \tag{2.14} $$

Now, we present a result which we think is new, and it will be very useful in the next section.

Lemma: *For the $q-$shifted factorial $(a;q)_n$ the following identity holds*

$$\sum_{k=0}^{v} q^{(y-w)k} = q^{(y-w)v} \frac{(q^{-y};q)_w}{(q^{1-y};q)_w (q^{-y};q)_1}. \tag{2.15}$$

Proof: We start with the identity

$$\sum_{k=0}^{v} q^{(y-w)k} = \sum_{k=0}^{v} \frac{(q;q)_k}{(q;q)_k} \frac{(q^{-v};q)_k}{(q^{-v};q)_k} q^{(y-w)k}$$

$$= \sum_{k=0}^{v} \frac{(q^{1-v}/q^{-v};q)_k}{(q;q)_k} \frac{(q^{-v};q)_k}{(q^{1-v}/q;q)_k} \left(\frac{q^{-v}q}{q^{-v}q}\right)^k q^{(y-w)k},$$

by reversing the order of summation by means of (2.11)

$$\sum_{k=0}^{v} q^{(y-w)k} = \frac{(q^{-v};q)_v}{(q;q)_v} \left(\frac{q}{q^{w-y}}\right)^k (-1)^v q^{\binom{v}{2}} \; {}_1\phi_0 \left(q;-;q,q^{w-y}\right).$$

Now, using the relation (2.7), we have

$$\frac{(q^{-v};q)_v}{(q;q)_v} = (-1)^v q^{-v} q^{-\binom{v}{2}}.$$

Therefore

$$\sum_{k=0}^{v} q^{(y-w)k} = q^{(y-w)v} \; {}_1\phi_0 \left(q;-;q,q^{w-y}\right)$$

$$= q^{(y-w)v} \frac{(q^{1+w-y};q)_\infty}{(q^{w-y};q)_\infty},$$

where we have applied (2.14).

This result can be written as

$$\sum_{k=0}^{v} q^{(y-w)k} = q^{(y-w)v} \frac{(q^{1-y};q)_\infty}{(q^{-y};q)_\infty} \frac{(q^{-y};q)_w}{(q^{1-y};q)_w},$$

and from (2.3) we get

$$\sum_{k=0}^{v} q^{(y-w)k} = q^{(y-w)v} \frac{(q^{-y};q)_w}{(q^{1-y};q)_w (q^{-y};q)_1}. \qquad \square$$

3. The operator $L_q^n(.)$

Now we define a new fractional q-integral operator, denoted by $L_q^n(.)$,which is introduced through the expression

$$L_{\boldsymbol{q}}^n\{M,b_1,b_2,...,b_r,\gamma,m_1,m_2,...,m_r;f(x)\} =$$

$$\frac{x^{-\gamma-1}}{\Gamma_q(M+1)} \int_0^x t^{\gamma} {}_{r+1}\phi_r \left[\begin{matrix} q^{-M}, q^{m_1+b_1}, ..., q^{m_r+b_r} \\ q^{b_1}, ..., q^{b_r} \end{matrix} ; q, q^n \frac{t}{x} \right] f(t)\; d_q t, \tag{3.1}$$

$M, m_1, ..., m_r$ nonnegative integers, $n \in \mathbb{N}, \gamma \in \mathbb{C},\ b_1, ..., b_r \neq 0, -1, -2, ..., \left|\frac{t}{x}\right| < 1,$

being [5, p. 16, No. (1.10.1)]

$$\begin{cases} \Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, & 0<q<1. \\ \Gamma_q(n) = (q;q)_{n-1}(1-q)^{1-n}, & 0<q<1. \end{cases} \tag{3.2}$$

By virtue of the result (2.8), (3.1) can be expressed as:

$$L_{\boldsymbol{q}}^{n}\{M,b_1,b_2,...,b_r,\rho,m_1,m_2,...,m_r;f(x)\}=\frac{(1-q)}{\Gamma_q(M+1)}\times$$

$$\sum_{k=0}^{\infty}q^{(\rho+1)k}\ {}_{r+1}\phi_r\left[\begin{array}{l}q^{-M},q^{b_1+m_1},...,q^{b_r+m_r}\\ q^{b_1},...,q^{b_r}\end{array};q,q^{n+k}\right]\ f(xq^k), \tag{3.3}$$

$M,m_1,...,m_r$ nonnegative integers, $n\in\mathbb{N},\rho\in\mathbb{C}$, $b_1,...,b_r\neq 0,-1,-2,...$

As particular cases of this operator we have:

i)Kalla operator:

$$\lim_{q\longrightarrow 1^-}L_{\boldsymbol{q}}^{n}\{M,b_1,b_2,...,b_r,\gamma,m_1,m_2,...,m_r;f(x)\}=$$

$$\frac{x^{-\gamma-1}}{\Gamma(M+1)}\int_0^x t^{\gamma}\ {}_{r+1}F_r\left[\begin{array}{l}-M,m_1+b_1,...,m_r+b_r\\ b_1,...,b_r\end{array};\frac{t}{x}\right]f(t)\ dt$$

$$=R^{1,r+1,r+1,r+1;-1}_{\gamma,1;1+M,1-(m_1+b_1),...,1-(m_r+b_r),0,1-b_1,...,1-b_r}[f(x)], \tag{3.4}$$

where $M>m_1+...+m_r$, and we have used (1.5).

ii) Delgado and Galué operator:

$$L_{\boldsymbol{q}}\{M,b_1,b_2,...,b_r,\gamma,m_1,m_2,...,m_r;f(x)\}=$$

$$L\{M,b_1,b_2,...,b_r,\gamma,m_1,m_2,...,m_r;f(x)\}, \tag{3.5}$$

with $L_{\boldsymbol{q}}^1\equiv L_{\boldsymbol{q}}$.

iii) Saxena operator:

$$\lim_{q\to 1^-}L_{\boldsymbol{q}}^{n}\{M,b_1,\gamma,m_1;f(x)\}=\mathfrak{T}\left[-M,b_1,\gamma,m_1;f(x)\right], \tag{3.6}$$

where $\gamma\in\mathbb{C}$, M, m_1 are nonnegative integers, $b_1\neq 0,-1,-2,...$

iv) Erdélyi Kober operator:

$$\lim_{q\to 1^-}L_{\boldsymbol{q}}^{n}\{M,b_1,\gamma,0;f(x)\}=\mathrm{I}_{\gamma,M+1}f(x), \tag{3.7}$$

with $\gamma\in\mathbb{C}$, M is nonnegative integer, $b_1\neq 0,-1,-2,...$

4. Composition formulae for $L_q^n(.)$ fractional $q-$integral operators

In this section we establish some composition formulae for the operators $L_q^n(.)$. For convenience we will use the following notation:

$$L_{\boldsymbol{q}}^{n}\{M,b_{11},b_{21},...,b_{r1},\rho,m_{11},m_{21},...,m_{r1};f(x)\}\equiv$$

$$L_{\boldsymbol{q}}^{n}\left[\begin{array}{l}M,b_{11},b_{21},...,b_{r1}\\ \rho,m_{11},m_{21},...,m_{r1}\end{array}\right]f(x).$$

Theorem 1: *Let be* $L_{\boldsymbol{q}}^{n}\left[\begin{array}{l}M,b_1,b_2,...,b_r\\ \rho,m_1,m_2,...,m_r\end{array}\right]f(x)$ *a fractional q-integral operator, as defined by (3.1), then*

$$L_{\boldsymbol{q}}\left[\begin{array}{l}M_2,b_{12},b_{22},...,b_{r2}\\ \rho+M_1+1,m_{12},m_{22},...,m_{r2}\end{array}\right]L_{\boldsymbol{q}}^{n}\left[\begin{array}{l}M_1,b_{11},b_{21},...,b_{r1}\\ \rho,m_{11},m_{21},...,m_{r1}\end{array}\right]f(x)=$$

$$\prod_{i=1}^{r}\left[\frac{\Gamma_q(b_{i2}+m_{i2}-1-M_1)}{\Gamma_q(b_{i2}-1-M_1)}\frac{\Gamma_q(b_{i2})}{\Gamma_q(b_{i2}+m_{i2})}\right]q^{(1+M_1)M_2}\quad\times$$

$$L_{\boldsymbol{q}}^{n+1}\left[\begin{array}{l} M_2+M_1+1, b_{11},...,b_{r1}, b_{12}-(M_1+1),...,b_{r2}-(M_1+1) \\ \rho, m_{11},...,m_{r1}, m_{12},...,m_{r2} \end{array}\right] f(x), \quad (4.1)$$

where $M_2 \geq m_{12}+...+m_{r2};\ b_{i2} \neq M_1+1, M_1, M_1-1,...$ *with* $i=1,...,r$.

Proof: Let be

$$L_{\boldsymbol{q}}^{n}\left[\begin{array}{l} M_1, b_{11}, b_{21},...,b_{r1} \\ \rho, m_{11}, m_{21},...,m_{r1} \end{array}\right] f(x)=g(x). \quad (4.2)$$

On applying (4.2) and (3.3) we obtain

$$L_{\boldsymbol{q}}\left[\begin{array}{l} M_2, b_{12}, b_{22},...,b_{r2} \\ \rho+M_1+1, m_{12}, m_{22},...,m_{r2} \end{array}\right] g(x)=$$

$$L_{\boldsymbol{q}}\left[\begin{array}{l} M_2, b_{12}, b_{22},...,b_{r2} \\ \rho+M_1+1, m_{12}, m_{22},...,m_{r2} \end{array}\right] L_{\boldsymbol{q}}^{n}\left[\begin{array}{l} M_1, b_{11}, b_{21},...,b_{r1} \\ \rho, m_{11}, m_{21},...,m_{r1} \end{array}\right] f(x)=$$

$$\frac{(1-q)^2}{\Gamma_q(M_1+1)\Gamma_q(M_2+1)}\sum_{v=0}^{\infty} q^{(\rho+M_1+2)v} f(xq^v) \quad \times$$

$$\sum_{k=0}^{v} q^{-(M_1+1)k}\ {}_{r+1}\phi_r\left[\begin{array}{l} q^{-M_2}, q^{b_{12}+m_{12}},...,q^{b_{r2}+m_{r2}} \\ q^{b_{12}},...,q^{b_{r2}} \end{array}; q, q^{1+v-k}\right] \quad \times$$

$${}_{r+1}\phi_r\left[\begin{array}{l} q^{-M_1}, q^{b_{11}+m_{11}},...,q^{b_{r1}+m_{r1}} \\ q^{b_{11}},...,q^{b_{r1}} \end{array}; q, q^{n+k}\right], \quad (4.3)$$

and using the result (2.10)

$$L_{\boldsymbol{q}}\left[\begin{array}{l} M_2, b_{12}, b_{22},...,b_{r2} \\ \rho+M_1+1, m_{12}, m_{22},...,m_{r2} \end{array}\right] g(x)=$$

$$\frac{(1-q)^2}{\Gamma_q(M_1+1)\Gamma_q(M_2+1)}\sum_{v=0}^{\infty} q^{(\rho+M_1+2)v} f(xq^v) \quad \times$$

$$\sum_{k=0}^{v} q^{-(M_1+1)k}\left\{\sum_{w=0}^{\infty}\frac{\left(q^{-M_2}, q^{b_{12}+m_{12}},...,q^{b_{r2}+m_{r2}}; q\right)_w}{(q, q^{b_{12}},...,q^{b_{r2}}; q)_w} q^{(1+v-k)w} \quad \times\right.$$

$$\left.\sum_{s=0}^{\infty}\frac{\left(q^{-M_1}, q^{b_{11}+m_{11}},...,q^{b_{r1}+m_{r1}}; q\right)_s}{(q, q^{b_{11}},...,q^{b_{r1}}; q)_s} q^{(n+k)s}\right\},$$

where [5, p. 6, No. 1.2.41]

$$(a_1, a_2,...,a_m; q)_n = (a_1; q)_n (a_2; q)_n ... (a_m; q)_n.$$

Interchanging the order of the sums we have

$$L_{\boldsymbol{q}}\left[\begin{array}{l} M_2, b_{12}, b_{22},...,b_{r2} \\ \rho+M_1+1, m_{12}, m_{22},...,m_{r2} \end{array}\right] g(x)=$$

$$\frac{(1-q)^2}{\Gamma_q(M_1+1)\Gamma_q(M_2+1)}\sum_{v=0}^{\infty} q^{(\rho+M_1+2)v} f(xq^v) \quad \times$$

$$\sum_{s=0}^{\infty}\frac{\left(q^{-M_1}, q^{b_{11}+m_{11}},...,q^{b_{r1}+m_{r1}}; q\right)_s}{(q, q^{b_{11}},...,q^{b_{r1}}; q)_s} q^{ns} \quad \times$$

$$\left\{\sum_{w=0}^{\infty}\frac{\left(q^{-M_2}, q^{b_{12}+m_{12}},...,q^{b_{r2}+m_{r2}}; q\right)_w}{(q, q^{b_{12}},...,q^{b_{r2}}; q)_w} q^{(1+v)w}\left[\sum_{k=0}^{v} q^{(s-w-M_1-1)k}\right]\right\}. \quad (4.4)$$

Since that by application of (2.15)

$$\sum_{k=0}^{v} q^{(s-w-M_1-1)k} = \frac{\left(q^{1-s+M_1};q\right)_w}{\left(q^{2-s+M_1};q\right)_w} \frac{q^{(s-w-M_1-1)v}}{\left(q^{1-s+M_1};q\right)_1}, \tag{4.5}$$

from (4.4) and (4.5) we obtain

$$L_{\boldsymbol{q}}\left[\begin{array}{l} M_2, b_{12}, b_{22}, ..., b_{r2} \\ \rho+M_1+1, m_{12}, m_{22}, ..., m_{r2} \end{array}\right] g(x)=$$

$$\frac{(1-q)^2}{\Gamma_q(M_1+1)\Gamma_q(M_2+1)} \sum_{v=0}^{\infty} q^{(\rho+1)v} f(xq^v) \quad \times$$

$$\sum_{s=0}^{\infty} \frac{\left(q^{-M_1}, q^{b_{11}+m_{11}}, ..., q^{b_{r1}+m_{r1}};q\right)_s}{(q, q^{b_{11}}, ..., q^{b_{r1}};q)_s} \frac{q^{(n+v)s}}{\left(q^{1-s+M_1};q\right)_1} \quad \times$$

$$\left\{ \sum_{w=0}^{\infty} \frac{\left(q^{-M_2}, q^{1-s+M_1}, q^{b_{12}+m_{12}}, ..., q^{b_{r2}+m_{r2}};q\right)_w}{(q, q^{2-s+M_1}, q^{b_{12}}, ..., q^{b_{r2}};q)_w} q^w \right\},$$

which using (2.10) can be written as

$$L_{\boldsymbol{q}}\left[\begin{array}{l} M_2, b_{12}, b_{22}, ..., b_{r2} \\ \rho+M_1+1, m_{12}, m_{22}, ..., m_{r2} \end{array}\right] g(x)=$$

$$\frac{(1-q)^2}{\Gamma_q(M_1+1)\Gamma_q(M_2+1)} \sum_{v=0}^{\infty} q^{(\rho+1)v} f(xq^v) \quad \times$$

$$\sum_{s=0}^{\infty} \frac{\left(q^{-M_1}, q^{b_{11}+m_{11}}, ..., q^{b_{r1}+m_{r1}};q\right)_s}{(q, q^{b_{11}}, ..., q^{b_{r1}};q)_s} \frac{q^{(n+v)s}}{\left(q^{1-s+M_1};q\right)_1} \quad \times$$

$${}_{r+2}\phi_{r+1}\left[\begin{array}{l} q^{-M_2}, q^{1-s+M_1}, q^{b_{12}+m_{12}}, ..., q^{b_{r2}+m_{r2}} \\ q^{2-s+M_1}, q^{b_{12}}, ..., q^{b_{r2}} \end{array} ;q,q\right]. \tag{4.6}$$

By virtue of the result (2.12) we get

$$L_{\boldsymbol{q}}\left[\begin{array}{l} M_2, b_{12}, b_{22}, ..., b_{r2} \\ \rho+M_1+1, m_{12}, m_{22}, ..., m_{r2} \end{array}\right] g(x)=$$

$$\frac{(1-q)^2}{\Gamma_q(M_1+1)\Gamma_q(M_2+1)} \sum_{v=0}^{\infty} q^{(\rho+1)v} f(xq^v) \quad \times$$

$$\sum_{s=0}^{\infty} \frac{\left(q^{-M_1}, q^{b_{11}+m_{11}}, ..., q^{b_{r1}+m_{r1}};q\right)_s}{(q, q^{b_{11}}, ..., q^{b_{r1}};q)_s} \frac{q^{(n+v)s}}{\left(q^{1-s+M_1};q\right)_1} \quad \times$$

$$\frac{q^{(1-s+M_1)M_2}(q;q)_{M_2}}{(q^{2-s+M_1};q)_{M_2}} \frac{(q^{b_{12}-1-M_1+s};q)_{m_{12}} ... (q^{b_{r2}-1-M_1+s};q)_{m_{r2}}}{(q^{b_{12}};q)_{m_{12}} ... (q^{b_{r2}};q)_{m_{r2}}} \tag{4.7}$$

with $M_2 \geq m_{12} + ... + m_{r2}$.

From (2.3) for $i = 1, 2, ..., r$

$$\frac{(q^{b_{i2}-1-M_1+s};q)_{m_{i2}}}{(q^{b_{i2}};q)_{m_{i2}}} = \frac{(q^{b_{i2}-1-M_1+s};q)_\infty}{(q^{b_{i2}+m_{i2}-1-M_1+s};q)_\infty} \frac{(q^{b_{i2}+m_{i2}};q)_\infty}{(q^{b_{i2}};q)_\infty} =$$

$$\frac{(q^{b_{i2}-1-M_1};q)_\infty}{(q^{b_{i2}+m_{i2}-1-M_1};q)_\infty} \frac{(q^{b_{i2}+m_{i2}};q)_\infty}{(q^{b_{i2}};q)_\infty} \frac{(q^{b_{i2}+m_{i2}-1-M_1};q)_s}{(q^{b_{i2}-1-M_1};q)_s},$$

now, applying (3.2) we have

$$\frac{(q^{b_{i2}-1-M_1+s};q)_{m_{i2}}}{(q^{b_{i2}};q)_{m_{i2}}}=\frac{\Gamma_q(b_{i2}+m_{i2}-1-M_1)}{\Gamma_q(b_{i2}-1-M_1)}\frac{\Gamma_q(b_{i2})}{\Gamma_q(b_{i2}+m_{i2})}\frac{(q^{b_{i2}+m_{i2}-1-M_1};q)_s}{(q^{b_{i2}-1-M_1};q)_s}.$$

On using this result and (2.4) in (4.7)

$$L_{\boldsymbol{q}}\left[\begin{array}{l} M_2,b_{12},b_{22},...,b_{r2}\\ \rho+M_1+1,m_{12},m_{22},...,m_{r2}\end{array}\right]g(x)=$$

$$\frac{(1-q)^2}{\Gamma_q(M_1+1)\Gamma_q(M_2+1)}\prod_{i=1}^{r}\left[\frac{\Gamma_q(b_{i2}+m_{i2}-1-M_1)}{\Gamma_q(b_{i2}-1-M_1)}\frac{\Gamma_q(b_{i2})}{\Gamma_q(b_{i2}+m_{i2})}\right]\quad\times$$

$$(q;q)_{M_2}\ q^{(1+M_1)M_2}\sum_{v=0}^{\infty}q^{(\rho+1)v}f(xq^v)\quad\times$$

$$\sum_{s=0}^{\infty}\frac{\left(q^{-M_1},q^{b_{11}+m_{11}},...,q^{b_{r1}+m_{r1}};q\right)_s}{(q,q^{b_{11}},...,q^{b_{r1}};q)_s}\quad\times$$

$$\frac{(q^{b_{12}+m_{12}-1-M_1},...,q^{b_{r2}+m_{r2}-1-M_1};q)_s}{(q^{b_{12}-1-M_1},...,q^{b_{r2}-1-M_1};q)_s}\frac{q^{(n-M_2+v)s}}{(q^{1-s+M_1};q)_{M_2+1}}.\tag{4.8}$$

On the other hand, applying the relation (2.6) we obtain

$$(q^{1-s+M_1};q)_{M_2+1}=\frac{(q^{1+M_1};q)_{M_2+1}(q^{-M_1};q)_s}{(q^{-M_2-M_1-1};q)_s}q^{-s(M_2+1)},\tag{4.9}$$

and from (2.5) with $a=q,\ n=M_2+M_1+1$ and $k=M_1$

$$(q^{1+M_1};q)_{M_2+1}=\frac{(q;q)_{M_2+M_1+1}}{(q;q)_{M_1}}.\tag{4.10}$$

Then from (4.9) and (4.10)

$$(q^{1-s+M_1};q)_{M_2+1}=\frac{(q;q)_{M_2+M_1+1}}{(q;q)_{M_1}}\frac{(q^{-M_1};q)_s}{(q^{-M_2-M_1-1};q)_s}q^{-s(M_2+1)}.$$

The substitution from this result in (4.8) leads us to

$$L_{\boldsymbol{q}}\left[\begin{array}{l} M_2,b_{12},b_{22},...,b_{r2}\\ \rho+M_1+1,m_{12},m_{22},...,m_{r2}\end{array}\right]g(x)=$$

$$\frac{(1-q)^2}{\Gamma_q(M_1+1)\Gamma_q(M_2+1)}\prod_{i=1}^{r}\left[\frac{\Gamma_q(b_{i2}+m_{i2}-1-M_1)}{\Gamma_q(b_{i2}-1-M_1)}\frac{\Gamma_q(b_{i2})}{\Gamma_q(b_{i2}+m_{i2})}\right]\quad\times$$

$$q^{(1+M_1)M_2}\frac{(q;q)_{M_1}(q;q)_{M_2}}{(q;q)_{M_2+M_1+1}}\sum_{v=0}^{\infty}q^{(\rho+1)v}f(xq^v)\quad\times$$

$$\sum_{s=0}^{\infty}\frac{\left(q^{-M_2-M_1-1},q^{b_{11}+m_{11}},...,q^{b_{r1}+m_{r1}};q\right)_s}{(q,q^{b_{11}},...,q^{b_{r1}};q)_s}\quad\times$$

$$\frac{(q^{b_{12}+m_{12}-1-M_1},...,q^{b_{r2}+m_{r2}-1-M_1};q)_s}{(q^{b_{12}-1-M_1},...,q^{b_{r2}-1-M_1};q)_s}q^{(n+1+v)s}.\tag{4.11}$$

Therefore from (2.10)

$$L_{\boldsymbol{q}}\left[\begin{array}{l} M_2,b_{12},b_{22},...,b_{r2}\\ \rho+M_1+1,m_{12},m_{22},...,m_{r2}\end{array}\right]g(x)=$$

$$\frac{(1-q)^2}{\Gamma_q(M_1+1)\Gamma_q(M_2+1)}\prod_{i=1}^{r}\left[\frac{\Gamma_q(b_{i2}+m_{i2}-1-M_1)}{\Gamma_q(b_{i2}-1-M_1)}\frac{\Gamma_q(b_{i2})}{\Gamma_q(b_{i2}+m_{i2})}\right]\quad\times$$

$$\frac{(q;q)_{M_1}(q;q)_{M_2}}{(q;q)_{M_2+M_1+1}}q^{(1+M_1)M_2}\sum_{v=0}^{\infty}q^{(\rho+1)v}f(xq^v) \quad \times$$

$${}_{2r+1}\phi_{2r}\left[\begin{array}{l} q^{-M_2-M_1-1},q^{b_{11}+m_{11}},...,q^{b_{r1}+m_{r1}},q^{b_{12}+m_{12}-1-M_1},...,q^{b_{r2}+m_{r2}-1-M_1} \\ q^{b_{11}},...,q^{b_{r1}},q^{b_{12}-1-M_1},...,q^{b_{r2}-1-M_1} \end{array};q,q^{n+1+v}\right]$$

$b_{i2}\neq M_1+1, M_1, M_1-1, ...$ with $i=1,...,r$.

Now, applying the definition (3.2) we obtain

$$L_{\boldsymbol{q}}\left[\begin{array}{l} M_2,b_{12},b_{22},...,b_{r2} \\ \rho+M_1+1,m_{12},m_{22},...,m_{r2}\end{array}\right]g(x)=$$

$$\prod_{i=1}^{r}\left[\frac{\Gamma_q(b_{i2}+m_{i2}-1-M_1)}{\Gamma_q(b_{i2}-1-M_1)}\frac{\Gamma_q(b_{i2})}{\Gamma_q(b_{i2}+m_{i2})}\right] \quad \times$$

$$\frac{(1-q)}{\Gamma_q(M_2+M_1+2)}q^{(1+M_1)M_2}\sum_{v=0}^{\infty}q^{(\rho+1)v}f(xq^v) \quad \times$$

$${}_{2r+1}\phi_{2r}\left[\begin{array}{l} q^{-M_2-M_1-1},q^{b_{11}+m_{11}},...,q^{b_{r1}+m_{r1}},q^{b_{12}+m_{12}-1-M_1},...,q^{b_{r2}+m_{r2}-1-M_1} \\ q^{b_{11}},...,q^{b_{r1}},q^{b_{12}-1-M_1},...,q^{b_{r2}-1-M_1} \end{array};q,q^{n+1+v}\right].$$

From this result and with the help from (3.3) and (4.2), the proof is complete. □

Theorem 2: *Let be* $L_{\boldsymbol{q}}^{n}\left[\begin{array}{l} M,b_1,b_2,...,b_r \\ \rho,m_1,m_2,...,m_r\end{array}\right]f(x)$ *a fractional q-integral operator, as defined by (3.1), then*

$$L_q\left[\begin{array}{l} M_j,b_{1j},b_{2j},...,b_{rj} \\ \rho+M_{j-1}+M_{j-2}+...+M_1+j-1,m_{1j},m_{2j},...,m_{rj}\end{array}\right]$$
$$L_q\left[\begin{array}{l} M_{j-1},b_{1j-1},b_{2j-1},...,b_{rj-1} \\ \rho+M_{j-2}+M_{j-3}+...+M_1+j-2,m_{1j-1},m_{2j-1},...,m_{rj-1}\end{array}\right]$$
$$...\ L_q\left[\begin{array}{l} M_2,b_{12},b_{22},...,b_{r2} \\ \rho+M_1+1,m_{12},m_{22},...,m_{r2}\end{array}\right]L_{\boldsymbol{q}}^{n}\left[\begin{array}{l} M_1,b_{11},b_{21},...,b_{r1} \\ \rho,m_{11},m_{21},...,m_{r1}\end{array}\right]f(x)=$$
$$\prod_{h=2}^{j}\prod_{i=1}^{r}\left[\frac{\Gamma_q\left(b_{ih}+m_{ih}-\left(\sum_{l=1}^{h-1}M_l+h-1\right)\right)}{\Gamma_q\left(b_{ih}-\left(\sum_{l=1}^{h-1}M_l+h-1\right)\right)}\frac{\Gamma_q(b_{ih})}{\Gamma_q(b_{ih}+m_{ih})}\right] \quad \times$$
$$q^{(M_{j-1}+...+M_1+j-1)M_j+...+(M_1+1)M_2} \quad \times$$
$$L_{\boldsymbol{q}}^{n+j-1}\left[\begin{array}{l} M_j+...+M_1+j-1,b_{11},...,b_{r1}, \\ \rho,m_{11},...,m_{r1}, \\ b_{12}-(M_1+1),...,b_{r2}-(M_1+1),..., \\ m_{12},...,m_{r2},..., \\ b_{1j-1}-(M_{j-2}+...+M_1+j-2),...,b_{rj-1}-(M_{j-2}+...+M_1+j-2), \\ m_{1j-1},...,m_{rj-1}, \\ b_{1j}-(M_{j-1}+...+M_1+j-1),...,b_{rj}-(M_{j-1}+...+M_1+j-1) \\ m_{1j},...,m_{rj}\end{array}\right]f(x), \tag{4.12}$$

where $M_j\geq m_{1j}+...+m_{rj}; b_{ij}\neq M_{j-1}+...+M_1+j-1, M_{j-1}+...+M_1+j-2,...$ *with* $i=1,...,r$ *and* $j=2,3,...$

Proof: To prove (4.12), we employ the mathematical induction principle.

We observe that for $j=2$ in (4.12), it yields to eq. (4.12) as:

$$L_{\boldsymbol{q}}\left[\begin{array}{l} M_2, b_{12}, b_{22}, ..., b_{r2} \\ \rho+M_1+1, m_{12}, m_{22}, ..., m_{r2} \end{array}\right] L_{\boldsymbol{q}}^{n}\left[\begin{array}{l} M_1, b_{11}, b_{21}, ..., b_{r1} \\ \rho, m_{11}, m_{21}, ..., m_{r1} \end{array}\right] f(x)=$$

$$\prod_{i=1}^{r}\left[\frac{\Gamma_q(b_{i2}+m_{i2}-1-M_1)}{\Gamma_q(b_{i2}-1-M_1)}\frac{\Gamma_q(b_{i2})}{\Gamma_q(b_{i2}+m_{i2})}\right] q^{(1+M_1)M_2} \quad \times$$

$$L_{\boldsymbol{q}}^{n+1}\left[\begin{array}{l} M_2+M_1+1, b_{11}, ..., b_{r1}, b_{12}-(M_1+1), ..., b_{r2}-(M_1+1) \\ \rho, m_{11}, ..., m_{r1}, m_{12}, ..., m_{r2} \end{array}\right] f(x). \quad (4.13)$$

We suppose that eq. (4.12) is true for $j=k-1>2$, that is,

$$L_q\left[\begin{array}{l} M_{k-1}, b_{1k-1}, b_{2k-1}, ..., b_{rk-1} \\ \rho+M_{k-2}+M_{k-3}+...+M_1+k-2, m_{1k-1}, m_{2k-1}, ..., m_{rk-1} \end{array}\right]$$
$$L_q\left[\begin{array}{l} M_{k-2}, b_{1k-2}, b_{2k-2}, ..., b_{rk-2} \\ \rho+M_{k-3}+M_{k-4}+...+M_1+k-3, m_{1k-2}, m_{2k-2}, ..., m_{rk-2} \end{array}\right]$$
$$...\ L_q\left[\begin{array}{l} M_2, b_{12}, b_{22}, ..., b_{r2} \\ \rho+M_1+1, m_{12}, m_{22}, ..., m_{r2} \end{array}\right] L_{\boldsymbol{q}}^{n}\left[\begin{array}{l} M_1, b_{11}, b_{21}, ..., b_{r1} \\ \rho, m_{11}, m_{21}, ..., m_{r1} \end{array}\right] f(x)=$$

$$\prod_{h=2}^{k-1}\prod_{i=1}^{r}\left[\frac{\Gamma_q\left(b_{ih}+m_{ih}-\left(\sum_{l=1}^{h-1}M_l+h-1\right)\right)}{\Gamma_q\left(b_{ih}-\left(\sum_{l=1}^{h-1}M_l+h-1\right)\right)}\frac{\Gamma_q(b_{ih})}{\Gamma_q(b_{ih}+m_{ih})}\right] \quad \times$$

$$q^{(M_{k-2}+...+M_1+k-2)M_{k-1}+...+(M_1+1)M_2} \quad \mathrm{x}$$

$$L_{\boldsymbol{q}}^{n+k-2}\left[\begin{array}{l} M_{k-1}+...+M_1+k-2, b_{11}, ..., b_{r1}, \\ \rho, m_{11}, ..., m_{r1}, \end{array}\right.$$
$$b_{12}-(M_1+1), ..., b_{r2}-(M_1+1), ...,$$
$$m_{12}, ..., m_{r2}, ...,$$
$$b_{1k-2}-(M_{k-3}+...+M_1+k-3), ..., b_{rk-2}-(M_{k-3}+...+M_1+k-3),$$
$$m_{1k-2}, ..., m_{rk-2},$$
$$\left.\begin{array}{l} b_{1k-1}-(M_{k-2}+...+M_1+k-2), ..., b_{rk-1}-(M_{k-2}+...+M_1+k-2) \\ m_{1k-1}, ..., m_{rk-1} \end{array}\right] f(x) \quad (4.14)$$

$M_{k-1}\geq m_{1k-1}+...+m_{rk-1}$; $b_{ik-1}\neq M_{k-2}+...+M_1+k-2, M_{k-2}+...+M_1+k-3, ...$ with $i=1,...,r$.

On operating both sides of the relation (4.14) by the operator $L_q\left[\begin{array}{l} M_k, b_{1k}, b_{2k}, ..., b_{rk} \\ \rho+M_{k-1}+M_{k-2}+...+M_1+k-1, m_{1k}, m_{2k}, ..., m_{rk} \end{array}\right](.)$, we obtain

$$L_q\left[\begin{array}{l} M_k, b_{1k}, b_{2k}, ..., b_{rk} \\ \rho+M_{k-1}+M_{k-2}+...+M_1+k-1, m_{1k}, m_{2k}, ..., m_{rk} \end{array}\right]$$
$$L_q\left[\begin{array}{l} M_{k-1}, b_{1k-1}, b_{2k-1}, ..., b_{rk-1} \\ \rho+M_{k-2}+M_{k-3}+...+M_1+k-2, m_{1k-1}, m_{2k-1}, ..., m_{rk-1} \end{array}\right]$$
$$L_q\left[\begin{array}{l} M_{k-2}, b_{1k-2}, b_{2k-2}, ..., b_{rk-2} \\ \rho+M_{k-3}+M_{k-4}+...+M_1+k-3, m_{1k-2}, m_{2k-2}, ..., m_{rk-2} \end{array}\right]$$
$$...\ L_q\left[\begin{array}{l} M_2, b_{12}, b_{22}, ..., b_{r2} \\ \rho+M_1+1, m_{12}, m_{22}, ..., m_{r2} \end{array}\right] L_{\boldsymbol{q}}^{n}\left[\begin{array}{l} M_1, b_{11}, b_{21}, ..., b_{r1} \\ \rho, m_{11}, m_{21}, ..., m_{r1} \end{array}\right] f(x)=$$

$$\prod_{h=2}^{k-1}\prod_{i=1}^{r}\left[\frac{\Gamma_q\left(b_{ih}+m_{ih}-\left(\sum_{l=1}^{h-1}M_l+h-1\right)\right)}{\Gamma_q\left(b_{ih}-\left(\sum_{l=1}^{h-1}M_l+h-1\right)\right)}\frac{\Gamma_q(b_{ih})}{\Gamma_q(b_{ih}+m_{ih})}\right]\times$$

$$q^{(M_{k-2}+...+M_1+k-2)M_{k-1}+...+(M_1+1)M_2}\times$$

$$L_q\left[\begin{array}{l} M_k, b_{1k}, b_{2k}, ..., b_{rk} \\ \rho+M_{k-1}+M_{k-2}+...+M_1+k-1, m_{1k}, m_{2k}, ..., m_{rk}\end{array}\right]$$

$$L_{\boldsymbol{q}}^{n+k-2}\left[\begin{array}{l} M_{k-1}+...+M_1+k-2, b_{11}, ..., b_{r1}, \\ \rho, m_{11}, ..., m_{r1}, \\ b_{12}-(M_1+1), ..., b_{r2}-(M_1+1), ..., \\ m_{12}, ..., m_{r2}, ..., \\ b_{1k-2}-(M_{k-3}+...+M_1+k-3), ..., b_{rk-2}-(M_{k-3}+...+M_1+k-3), \\ m_{1k-2}, ..., m_{rk-2}, \\ b_{1k-1}-(M_{k-2}+...+M_1+k-2), ..., b_{rk-1}-(M_{k-2}+...+M_1+k-2) \\ m_{1k-1}, ..., m_{rk-1}\end{array}\right] f(x).$$

On applying (4.13) the right hand side of the above expression can be written as:

$$=\prod_{h=2}^{k-1}\prod_{i=1}^{r}\left[\frac{\Gamma_q\left(b_{ih}+m_{ih}-\left(\sum_{l=1}^{h-1}M_l+h-1\right)\right)}{\Gamma_q\left(b_{ih}-\left(\sum_{l=1}^{h-1}M_l+h-1\right)\right)}\frac{\Gamma_q(b_{ih})}{\Gamma_q(b_{ih}+m_{ih})}\right]\times$$

$$\prod_{i=1}^{r}\left[\frac{\Gamma_q(b_{ik}+m_{ik}-(M_{k-1}+...+M_1+k-1))}{\Gamma_q(b_{ik}-(M_{k-1}+...+M_1+k-1))}\frac{\Gamma_q(b_{ik})}{\Gamma_q(b_{ik}+m_{ik})}\right]\times$$

$$q^{(M_{k-1}+...+M_1+k-1)M_k+(M_{k-2}+...+M_1+k-2)M_{k-1}+...+(M_1+1)M_2}\times$$

$$L_{\boldsymbol{q}}^{n+k-1}\left[\begin{array}{l} M_k+...+M_1+k-1, b_{11}, ..., b_{r1}, \\ \rho, m_{11}, ..., m_{r1}, \\ b_{12}-(M_1+1), ..., b_{r2}-(M_1+1), ..., \\ m_{12}, ..., m_{r2}, ..., \\ b_{1k-2}-(M_{k-3}+...+M_1+k-3), ..., b_{rk-2}-(M_{k-3}+...+M_1+k-3), \\ m_{1k-2}, ..., m_{rk-2}, \\ b_{1k-1}-(M_{k-2}+...+M_1+k-2), ..., b_{rk-1}-(M_{k-2}+...+M_1+k-2) \\ m_{1k-1}, ..., m_{rk-1} \\ b_{1k}-(M_{k-1}+...+M_1+k-1), ..., b_{rk}-(M_{k-1}+...+M_1+k-1) \\ m_{1k}, ..., m_{rk}\end{array}\right] f(x)$$

$M_k \geq m_{1k}+...+m_{rk}$; $b_{ik} \neq M_{k-1}+...+M_1+k-1, M_{k-1}+...+M_1+k-2, ...$ with $i=1,...,r$.

Which is true for $j=k$. This completes the proof of (4.12). □

The Theorem 2 contains some interesting corollaries. Setting $n=1$, we obtain

Corollary 1: *Let be* $L\left[\begin{array}{l} M, b_1, b_2, ..., b_r \\ \rho, m_1, m_2, ..., m_r\end{array}\right] f(x)$ *a Delgado and Galué operator, as defined by (1.8), then*

$$L\left[\begin{array}{l} M_j, b_{1j}, b_{2j}, ..., b_{rj} \\ \rho + M_{j-1} + M_{j-2} + ... + M_1 + j - 1, m_{1j}, m_{2j}, ..., m_{rj} \end{array}\right]$$
$$L\left[\begin{array}{l} M_{j-1}, b_{1j-1}, b_{2j-1}, ..., b_{rj-1} \\ \rho + M_{j-2} + M_{j-3} + ... + M_1 + j - 2, m_{1j-1}, m_{2j-1}, ..., m_{rj-1} \end{array}\right]$$
$$... \, L\left[\begin{array}{l} M_2, b_{12}, b_{22}, ..., b_{r2} \\ \rho + M_1 + 1, m_{12}, m_{22}, ..., m_{r2} \end{array}\right] L\left[\begin{array}{l} M_1, b_{11}, b_{21}, ..., b_{r1} \\ \rho, m_{11}, m_{21}, ..., m_{r1} \end{array}\right] f(x) =$$
$$\prod_{h=2}^{j}\prod_{i=1}^{r}\left[\frac{\Gamma_q\left(b_{ih} + m_{ih} - \left(\sum_{l=1}^{h-1} M_l + h - 1\right)\right)}{\Gamma_q\left(b_{ih} - \left(\sum_{l=1}^{h-1} M_l + h - 1\right)\right)} \frac{\Gamma_q(b_{ih})}{\Gamma_q(b_{ih} + m_{ih})}\right] \times$$
$$q^{(M_{j-1}+...+M_1+j-1)M_j+...+(M_1+1)M_2} \times$$
$$L_{\boldsymbol{q}}^{j}\left[\begin{array}{l} M_j + ... + M_1 + j - 1, b_{11}, ..., b_{r1}, \\ \rho, m_{11}, ..., m_{r1}, \\ b_{12} - (M_1+1), ..., b_{r2} - (M_1+1), ..., \\ m_{12}, ..., m_{r2}, ..., \\ b_{1j-1} - (M_{j-2} + ... + M_1 + j - 2), ..., b_{rj-1} - (M_{j-2} + ... + M_1 + j - 2), \\ m_{1j-1}, ..., m_{rj-1}, \\ b_{1j} - (M_{j-1} + ... + M_1 + j - 1), ..., b_{rj} - (M_{j-1} + ... + M_1 + j - 1) \\ m_{1j}, ..., m_{rj} \end{array}\right] f(x), \tag{4.15}$$

where $M_j \geq m_{1j} + ... + m_{rj}; b_{ij} \neq M_{j-1} + ... + M_1 + j - 1, M_{j-1} + ... + M_1 + j - 2, ...$ *with* $i = 1, ..., r$ *and* $j = 2, 3, ...$

Taking $r = 1$ and $q \to 1^-$, we have

Corollary 2: *Let be* $\mathfrak{T}[-M, b_1, \rho, m_1] f(x)$ *a Saxena operator, as defined by (1.3), then*

$$\mathfrak{T}[-M_j, b_{1j}, \rho + M_{j-1} + M_{j-2} + ... + M_1 + j - 1, m_{1j}]$$
$$\mathfrak{T}[-M_{j-1}, b_{1j-1}, \rho + M_{j-2} + M_{j-3} + ... + M_1 + j - 2, m_{1j-1}]$$
$$... \, \mathfrak{T}[-M_2, b_{12}, \rho + M_1 + 1, m_{12}] \, \mathfrak{T}[-M_1, b_{11}, \rho, m_{11}] f(x) =$$
$$\prod_{h=2}^{j}\left[\frac{\Gamma\left(b_{1h} + m_{1h} - \left(\sum_{l=1}^{h-1} M_l + h - 1\right)\right)}{\Gamma\left(b_{1h} - \left(\sum_{l=1}^{h-1} M_l + h - 1\right)\right)} \frac{\Gamma(b_{1h})}{\Gamma(b_{1h} + m_{1h})}\right] \times$$
$$R^{1,j+1,j+1,j+1;-1}_{\gamma,1;M_j+M_{j-1}+...+M_1+j,1-(m_{11}+b_{11}),a_j,0,1-b_{11},b_j}[f(x)], \tag{4.16}$$

where $a_j = 1 - (m_{1j} + b_{1j} - (M_{j-1} + ... + M_1 + j - 1))$, $b_j = 1 - b_{1j} + (M_{j-1} + ... + M_1 + j - 1)$ *and* $M_j > m_{1j}; b_{1j} \neq M_{j-1} + ... + M_1 + j - 1, M_{j-1} + ... + M_1 + j - 2, ...$ *with* $j = 2, 3, ...$

Putting $r = 1, m_{11} = m_{12} = \cdots = m_{1j} = 0$ and $q \to 1^-$, we get

Corollary 3: *Let be* $\mathrm{I}_{\rho,M+1} f(x)$ *an Erdélyi-Kober operator, as defined by (1.2), then*

$$\mathrm{I}_{\rho+M_{j-1}+M_{j-2}+...+M_1+j-1,M_j+1}\mathrm{I}_{\rho+M_{j-2}+M_{j-3}+...+M_1+j-2,M_{j-1}+1}$$

$$... \mathrm{I}_{\rho+M_1+1,M_2+1}\mathrm{I}_{\rho,M_1+1}f(x)=\mathrm{I}_{\rho,M_j+...+M_1+j}\ f(x), \quad j=2,3,... \tag{4.17}$$

5. References

[1] R. P. Agarwal, *Certain fractional q-integral and q-derivatives.* Math. Proc. Camb. Phil. Soc. 66 (1969), 365-370.

[2] M. Delgado and L. Galué, *Fractional q-integral operator involving basic hypergeometric function.* Algebras Groups Geom. 25 (2008), 53-74.

[3] L. Galué, *Further results on fractional calculus operators involving the Appell function F_3.* Kuwait J. Sci. Engrg. 30 (1), (2003), 1-13.

[4] G. Gasper, *q-Series.* Lecture Notes by George Gasper, Department of Math. Northwestern University, Evaston, IL 60208, 1995.

[5] G. Gasper and M. Rahman, *Basic Hypergeometric Series.* Encyclopedia of Mathematics and its Applications, Vol. 35, Cambridge University Press, 1990.

[6] R.N. Kalia (Ed), *Recent Advances in Fractional Calculus.* Global Publishing Company, U.S.A., 1993.

[7] S.L. Kalla, *Fractional integration operators involving generalized hypergeometric functions.* Univ. Nac. Tucumán, Rev. Matemática y Física Teórica XX (1970), 93-100.

[8] S.L. Kalla, L. Galué and H.M. Srivastava, *Further results on an H-function generalized fractional calculus.* J. Fract. Calc. 4 (1993), 89-102.

[9] S. L. Kalla and V. Kiryakova, *An H-function generalized fractional calculus based upon compositions of Erdélyi-Kober operators in L_p.* Math. Japon. 35 (1990), 1151-1171.

[10] V. Kiryakova, *Generalized Fractional Calculus and Applications.* Longman (Pitman, Res. Notes in Math. Series # 301), London, U.K., 1994.

[11] A.C. McBride and G.F. Roach (Eds),. *Fractional Calculus.* Res. Notes in Math., Pitman 138, 1985.

[12] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series: More Special Functions*, Vol. 3. Gordon and Breach Science Publishers, New York, 1990.

[13] B. Ross (Ed), *Fractional Calculus and Its Applications.* Lecture Notes in Math., Springer-Verlag 457, 1975.

[14] M. Saigo, *A remark on integral operators involving the Gauss hypergeometric functions.* Kyushu University Mathematics Reports of the College of General Education XI (2) (1978), 135-143.

[15] M. Saigo, *On generalized fractional calculus operators.* Proceedings of International Workshop on the Recent Advances in Applied Mathematics-1996, State of Kuwait (1996), 441-449.

[16] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives.* Gordon and Breach Science Publishers, New York, 1993.

[17] R.K. Saxena, *On fractional integration operators.* Math. Zeitschr. 96 (4) (1967), 288-291.

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