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LOG-CONVEXITY OF RATIO OF THE TWO-PARAMETER SYMMETRIC HOMOGENEOUS FUNCTIONS AND AN APPLICATION

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Abstract. In this paper, four types of log-convexity of ratio of the two-parameter symmetric homogeneous functions are investigated and a Hermite-Hadamard type inequality is established. As an application, the log-convexity of ratio of Stolarsky means and a Hermite-Hadamard type inequality are presented. Additionally, some classical and new inequalities of ratio of bivariate means are illustrated.

1. Introduction

Since Ky Fan [1] inequality was presented, inequalities of ratio of means have attracted attentions of many scholars. Some known results can be found in [21, 3, 10, 14]. Research for the properties of ratio of bivariate means were also a hotspot at one time. In 1994 Pearce and C. E. M. Pečarić [15] proved that the function

$$p \to \frac{L_p(a,b)}{L_p(c,d)} \quad (p \in \mathbb{R})$$

is nondecreasing, provided that $a, b, c, d > 0$ with $b/a \geq d/c$. Here $L_p(a,b) := S_{p+1,1}(a,b)$ is the generalized logarithmic mean and $S_{p,q}(a,b)$ are the Stolarsky means of $a, b > 0$ with parameters $p, q \in \mathbb{R}$ [19] defined by

$$S_{p,q}(a,b) = \begin{cases} \left(\dfrac{q(a^p - b^p)}{p(a^q - b^q)}\right)^{\frac{1}{p-q}}, & p \neq q, pq \neq 0; \\ \left(\dfrac{a^p - b^p}{p(\ln a - \ln b)}\right)^{\frac{1}{p}}, & p \neq 0, q = 0; \\ \left(\dfrac{a^q - b^q}{q(\ln a - \ln b)}\right)^{\frac{1}{q}}, & p = 0, q \neq 0; \\ \exp\left(\dfrac{a^p \ln a - b^p \ln b}{a^p - b^p} - \dfrac{1}{p}\right), & p = q \neq 0; \\ \sqrt{ab}, & p = q = 0, \end{cases} \tag{1.1}$$

and $S_{p,q}(a,a) = a$. In a few years, Ch.-P. Chen and F. Qi [4], [17],[18], [5] also proved equivalent results.

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Recently Ch.-P. Chen [6, 7] established a more general result: Let a, b, c, d be fixed positive numbers with $a \neq b, c \neq d$ and p, q be real numbers. Then the function

$$R_{p,q} = R_{p,q}(a, b; c, d) := \frac{S_{p,q}(a, b)}{S_{p,q}(c, d)} \tag{1.2}$$

is increasing (decreasing) with both p and q according as $\min(a, b)/\max(a, b) \geq (\leq) \min(c, d)/\max(c, d)$. Soon after L. Losonczi [13] studied monotonicity properties of the ratio

$$R_{p,q}(a, b, c) := \frac{S_{p,q}(a, b)}{S_{p,q}(a, c)} \quad (p, q \in \mathbb{R}, 0 < a < b < c)$$

in the parameters p, q and completely solve the comparison problem

$$R_{p,q}(a, b, c) \leq R_{r,s}(a, b, c) \quad (p, q, r, s \in \mathbb{R}, 0 < a < b < c)$$

for this ratio. This generalizes all results mentioned above. Also, an open problem was proposed by the author: Let $M_{p,q}(p, q \in \mathbb{R})$ be a two-parameter, symmetric and homogeneous mean defined for positive variables and let us form the ratio

$$R_{p,q}(a, b, c) := \frac{M_{p,q}(a, b)}{M_{p,q}(a, c)} \quad (p, q \in \mathbb{R}, 0 < a < b < c).$$

For what means $M_{p,q}$ has this ratio simple monotonicity properties?

Z. -H. Yang [26] has Considered the ratio defined by

$$R_f(p, q) := \frac{\mathcal{H}_f(p, q; a, b)}{\mathcal{H}_f(p, q; c, d)} \quad (p, q \in \mathbb{R}, a, b, c, d > 0 \text{ with } b/a > d/c \geq 1\ ) \tag{1.3}$$

and presented four types of monotonicity of $R_f(p, q)$ in the parameters p and q, which gave an easier access to find two-parameter symmetric homogeneous means having ratio simple monotonicity properties mentioned by L. Losonczi.

Here $\mathcal{H}_f(p, q; a, b)$ are the so-called two-parameter homogeneous functions [23]. For conveniences, we record it as follows.

Definition 1.1. *Assume $f\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is n-order homogeneous and continuous, and has first partial derivatives and $(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+$, $(p, q) \in \mathbb{R} \times \mathbb{R}$. Then define*

$$\mathcal{H}_f(p, q; a, b) = \left(\frac{f(a^p, b^p)}{f(a^q, b^q)}\right)^{1(p-q)} \quad \text{if } p \neq q, \tag{1.4}$$

$$\mathcal{H}_f(p, p; a, b) = \exp\left(\frac{a^p f_x(a^p, b^p) \ln a + b^p f_y(a^p, b^p) \ln b}{f(a^p, b^p)}\right) \quad \text{if } p = q, \tag{1.5}$$

where $f_x(x, y)$ and $f_y(x, y)$ denote first-order partial derivatives with respect to first and second component of $f(x, y)$, respectively.

Since f is homogeneous and so are $\mathcal{H}_f(p, q; a, b)$ for every p, q. It is called a homogeneous function with parameters p and q, and usually simply denotes by $\mathcal{H}_f(p, q)$ or $\mathcal{H}_f(a, b)$ or $\mathcal{H}_f$.

The aim of this paper is to investigate the log-convexity of ratio of the two-parameter homogeneous functions in the parameters p, q .

The organization of the paper is as follows. In section 2, some properties of the two-parameter homogeneous functions are listed. Four types of log-convexity ratio of two-parameter homogeneous functions are investigated in section 3. In section 4, as an application, the log-convexity of ratio of Stolarsky means are given.

2. LEMAAS

To formulate main results, we need some properties of the two-parameter homogeneous functions.

Propority 2.1. *$\mathcal{H}_f(p,q)$ is symmetric with respect to p,q, i.e.,*

$$\mathcal{H}_f(p,q)=\mathcal{H}_f(q,p). \tag{2.1}$$

Propority 2.2. *If $f(x,y)$ is symmetric and n-order homogeneous with respect to x and y, then*

$$\mathcal{H}_f(-p,-q;a,b) = \frac{G^{2n}}{\mathcal{H}_f(p,q;a,b)}, \tag{2.2}$$

$$\mathcal{H}_f(p,-p;a,b) = G^n, \tag{2.3}$$

where $G=\sqrt{ab}$.

Denote by

$$T(t)=T(t;a,b):=\ln f(a^t,b^t), \tag{2.4}$$

which is useful in the sequel.

Lemma 2.3 ([24, (1.14), (1.15), (2.10), (2.11)]). *Let $f:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ be a symmetric, n-order homogenous and three-time differentiable function. Then we have*

$$T(t)-T(-t) = 2nt\ln G, \tag{2.5}$$

$$T'(t)+T'(-t) = 2n\ln G=2T'(0), \tag{2.6}$$

$$T''(-t) = T''(t), \tag{2.7}$$

$$T'''(-t) = -T'''(t), \tag{2.8}$$

where $G=\sqrt{ab}$.

Lemma 2.4 ([24, Lemma 3, 4]). *Let $f:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ be a symmetric, homogenous and three-time differentiable function. Then*

$$T'(t) = \frac{a^tf_x(a^t,b^t)\ln a+b^tf_y(a^t,b^t)\ln b}{f(a^t,b^t)}=\mathcal{H}_f(t,t;a,b), \tag{2.9}$$

$$T''(t) = -xy\mathcal{I}\ln^2(b/a),\quad \mathcal{I}=(\ln f)_{xy}, \tag{2.10}$$

$$T'''(t) = -xy(x\mathcal{I})_x\ln^3(a/b), \tag{2.11}$$

where $x=a^t, y=b^t$.

Lemma 2.5 ([26, Remark 2.7]). *If $T'(t)$ is continuous on $[p,q]$ or $[q,p]$, then $\ln\mathcal{H}_f(p,q)$ can be expressed in integral form as*

$$\ln\mathcal{H}_f(p,q)=\begin{cases}\dfrac{1}{p-q}\int_q^pT'(t)dt & \text{if } p\neq q\\ T'(q) & \text{if } p=q\end{cases}=\int_0^1T'(tp+(1-t)q)dt. \tag{2.12}$$

The following lemma will be used in the proof of second and fourth log-convexity.

Lemma 2.6 ([26, Lemma 3.3]). *Let $x \to f(x)$ be an odd and continuous function defined on $[-m, m]$ $(m > 0)$. Then*

$$\int_s^r f(x)dx = \int_{|s|}^{|r|} f(x)dx \tag{2.13}$$

is true for arbitrary $r, s \in [-m, m]$.

To prove Hermite-Hadamard type inequality, we need the extension of the Hermite-Hadamard inequality given by P. Czinder and Zs. Páles [9, Theorem 2.2].

Lemma 2.7 ([9, Theorem 2.2]). *Let $f : \mathbb{J} \to \mathbb{R}$ be symmetric with respect to an element $m \in \bar{\mathbb{J}}$. Furthermore, suppose that f is convex over the interval $\mathbb{J} \cap (-\infty, m]$ and concave over $\mathbb{J} \cap [m, +\infty)$. Then, for any interval $[q, p] \subset \mathbb{J}$*

$$f(\frac{p+q}{2}) \leq (\geq) \frac{1}{p-q} \int_q^p f(t)dt \leq (\geq) \frac{f(p)+f(q)}{2} \tag{2.14}$$

holds if $\frac{p+q}{2} \leq (\geq) m$.

In (2.14) the reversed inequalities are valid if f is concave over the interval $\mathbb{J} \cap [-\infty, m)$ and convex over $\mathbb{J} \cap [m, +\infty)$.

3. Main Results

Based on the above properties and lemmas, Yang has investigated the monotonicity and log-convexity of two-parameter homogeneous functions and obtained a series of valuable results in [23, 24], which yield some new and interesting inequalities for means. Recently, two results on monotonicity and log-convexity of a four-parameter homogeneous containing Stolarsky mean and Gini mean are presented in [25].

In these proving processes on [23, 24, 25], two decision functions play an important role, that are:

$$\mathcal{I} = \mathcal{I}(x,y) = \frac{\partial^2 \ln f(x,y)}{\partial x \partial y} = (\ln f(x,y))_{xy} = (\ln f)_{xy}, \tag{3.1}$$

$$\mathcal{J} = \mathcal{J}(x,y) = (x-y)\frac{\partial(x\mathcal{I})}{\partial x} = (x-y)(x\mathcal{I})_x. \tag{3.2}$$

In what follows we will encounter another key decision function defined by

$$\mathcal{T}_3(x,y) := -xy(x\mathcal{I})_x \ln^3(x/y), \text{ where } \mathcal{I} = (\ln f)_{xy}, \, x = a^t, y = b^t. \tag{3.3}$$

Comparing (2.11) with (3.3), we see that $T'''(t)$ and $\mathcal{T}_3(x,y)$ have the following relation:

$$T'''(t) = t^{-3}\mathcal{T}_3(x,y), \text{ where } x = a^t, y = b^t. \tag{3.4}$$

Moreover, It is easy to verify that $\mathcal{T}_3(x,y)$ is zero-order homogeneous due to homogeneity of $f(x,y)$, thus

$$\mathcal{T}_3(x,y) = \mathcal{T}_3(x/y, 1) = \mathcal{T}_3(1, y/x). \tag{3.5}$$

Next let us consider the log-convexity of ratio of two-parameter homogeneous functions defined by (1.3).

To begin with, we consider the log-convexity of $R_f(p,q)$ defined by (1.3) in the special case of $q = p$.

Theorem 3.1. *Suppose that $f\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a homogenous and three-time differentiable function and $\mathcal{T}_3(1,u)$ strictly increase (decrease) with $u>1$ and decrease (increase) with $0<u<1$. Then for any $a,b,c,d>0$ with $b/a > d/c \geq 1$, $R_f(p,p)$ is strictly log-convex (log-concave) on $(0,\infty)$ and log-concave (log-convex) on $(-\infty,0)$.*

Proof. By (2.9) we see that

$$\ln \mathcal{H}_f(p,p;a,b) = \frac{\partial T(p;a,b)}{\partial p} \triangleq T'(p;a,b).$$

Two times derivation calculations lead to

$$\frac{\partial^2 \ln \mathcal{H}_f(p,p;a,b)}{\partial p^2} = \frac{\partial^3 T(t;a,b)}{\partial t^3} \triangleq T'''(p;a,b).$$

From (1.3) we have

$$\begin{aligned} \frac{d^2 \ln R_f(p,p)}{dp^2} &= \frac{\partial^2 \ln \mathcal{H}_f(p,p;a,b)}{\partial p^2} - \frac{\partial^2 \ln \mathcal{H}_f(p,p;c,d)}{\partial p^2} \\ &= T'''(p;a,b) - T'''(p;c,d). \end{aligned}$$

since $\mathcal{T}_3(1,u)$ strictly increase (decrease) with $u>1$ and decrease (increase) with $0<u<1$, (3.4) together with $b/a > d/c \geq 1$ yields

$$\begin{aligned} T'''(t;a,b) - T'''(t;c,d) &= t^{-3}\left(\mathcal{T}_3(a^t,b^t) - \mathcal{T}_3(c^t,d^t)\right) \\ &= t^{-3}\left(\mathcal{T}_3(1,(b/a)^t) - \mathcal{T}_3(1,(d/c)^t)\right) \begin{cases} >(<)0 \text{ for } t>0, \\ <(>)0 \text{ for } t<0. \end{cases} \end{aligned} \tag{3.6}$$

This means that

$$\frac{d^2 \ln R_f(p,p)}{dp^2} \begin{cases} >(<)0 \text{ for } p>0, \\ <(>)0 \text{ for } p<0. \end{cases}$$

The proof ends. □

We continue to deal with the log-convexity of $R_f(p,q)$.

Theorem 3.2 (first log-convexity)**.** *Suppose that $f\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a symmetric, n-order homogenous and three-time differentiable function and $\mathcal{T}_3(1,u)$ strictly increase (decrease) with $u>1$ and decrease (increase) with $0<u<1$. Then for any $a,b,c,d>0$ with $b/a > d/c \geq 1$ and fixed $q \in \mathbb{R}$, $R_f(p,q)$ is strictly log-convex (log-concave) in p on $(\frac{|q|-q}{2},\infty)$ and log-concave (log-convex) on $(-\infty,-\frac{|q|+q}{2})$.*

Proof. We first prove that $R_f(p,q)$ is strictly log-convex (log-concave) in p on $(\frac{|q|-q}{2},\infty)$. This can be divided two cases.

1) Case 1: $p,q>0$.

Two times partial derivative calculations for (2.12) lead to

$$\frac{\partial^2 \ln \mathcal{H}_f(p,q)}{\partial p^2} = \int_0^1 t^2 T'''(tp+(1-t)q)dt. \tag{3.7}$$

From $\ln R_f(p,q) = \ln \mathcal{H}_f(p,q;a,b) - \ln \mathcal{H}_f(p,q;c,d)$, we have

$$\frac{\partial^2 \ln R_f(p,q)}{\partial p^2} = \int_0^1 t^2 \left(T'''(tp+(1-t)q;a,b) - T'''(tp+(1-t)q;c,d)\right)dt. \tag{3.8}$$

From (3.6) it follows that

$$\frac{\partial^2 \ln R_f(p,q)}{\partial p^2} \begin{cases} > (<)0 \text{ if } p,q > 0, \\ < (>)0 \text{ if } p,q < 0. \end{cases}$$

2) Case 2: $p > -q \geq 0$.
Substituting $v = tp + (1-t)q$ in the integral of right hand side of (3.7) yields

$$\frac{\partial^2 \ln \mathcal{H}_f(p,q)}{\partial p^2} = \frac{1}{(p-q)^3} \int_q^p (v-q)^2 T'''(v)dv, \tag{3.9}$$

and then

$$\frac{\partial^2 \ln R_f(p,q)}{\partial p^2} = \frac{1}{(p-q)^3} \int_q^p (t-q)^2 \left(T'''(t;a,b) - T'''(t;c,d)\right) dt. \tag{3.10}$$

Since $\chi_1 := (p-q)^{-3} > 0$, it suffices to prove

$$\chi_2 := \int_q^p (t-q)^2 \left(T'''(t;a,b) - T'''(t;c,d)\right) dt > (<)0 \text{ for } p > -q \geq 0.$$

For this purpose, we split the integral $\int_q^p (t-q)^2 T'''(t;a,b)dt$ into two parts:

$$\int_q^p (\cdots)\, dt = \int_q^{-q} (\cdots)\, dt + \int_{-q}^p (\cdots)\, dt.$$

Noting the function $t \to T'''(t;a,b)$ is odd and so is the function $t \to (t^2 + q^2)T'''(t;a,b)$ due to (2.8), using properties of definite integral of the even and odd functions on symmetric bounded interval, the first part is equal to

$$\begin{aligned} \int_q^{-q} (t-q)^2 T'''(t;a,b)dt &= \int_q^{-q} (t^2 + q^2 - 2tq) T'''(t;a,b)dt \\ &= \int_q^{-q} (t^2+q^2) T'''(t;a,b)dt - 2q \int_q^{-q} tT'''(t;a,b)dt \\ &= -4q \int_0^{-q} tT'''(t;a,b)dt, \end{aligned}$$

thus,

$$\begin{aligned} \int_q^p (t-q)^2 T'''(t;a,b)dt &= \int_q^{-q} (t-q)^2 T'''(t;a,b)dt + \int_{-q}^p (t-q)^2 T'''(t;a,b)dt \\ &= -4q \int_0^{-q} tT'''(t;a,b)dt + \int_{-q}^p (t-q)^2 T'''(t;a,b)dt. \end{aligned}$$

Hence,

$$\begin{aligned} \chi_2 &= -4q \int_0^{-q} t \left(T'''(t;a,b) - T'''(t;c,d)\right) dt \\ &\quad + \int_{-q}^p (t-q)^2 \left(T'''(t;a,b) - T'''(t;c,d)\right) dt \text{ for } p > -q \geq 0. \end{aligned}$$

It follows from (3.6) and $q \leq 0$ that $\chi_2 > (<)0$. Which together with $\chi_1 = (p-q)^{-3} > 0$ yields (3.10)

$$\frac{\partial^2 \ln R_f(p,q)}{\partial p^2} = \chi_1 \cdot \chi_2 > (<)0 \text{ if } p > -q \geq 0.$$

The Case 1 and Case 2 show that function $R_f(p,q)$ is strictly log-convex (log-concave) in p if $p,q>0$ or $p>-q\geq 0$.

For $p,q<0$ or $p<-q\leq 0$. (2.2) yields

$$R_f(p,q)=\frac{G^{2n}(a,b)}{G^{2n}(c,d)}\frac{1}{R_f(-p,-q)}.$$

From the log-convexity of the function $R_f(p,q)$ in the cases of $p,q>0$ and $p>-q\geq 0$ it follows that $R_f(p,q)$ is strictly log-concave (log-convex) in p if $p,q<0$ or $p<-q\leq 0$.

This completes the proof. □

Remark 3.3. *It is clear that Theorem 3.2 generalizes and improves main result given in* [24, Theorem 1].

Theorem 3.4 (second log-convexity)**.** *Suppose that $f\colon \mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is a symmetric, homogenous and three-time differentiable function and $\mathcal{T}_3(1,u)$ strictly increase (decrease) with $u>1$ and decrease (increase) with $0<u<1$. Then for any $a,b,c,d>0$ with $b/a>d/c\geq 1$ and fixed $m>0$, $R_f(p,p+m)$ is strictly log-convex (log-concave) in p on $(-m/2,\infty)$ and log-concave (log-convex) in p on $(-\infty,-m/2)$.*

Proof. By (2.12) $\ln\mathcal{H}_f(p,p+m)$ can be expressed in integral form as

$$\ln\mathcal{H}_f(p,p+m)=\int_0^1 T(tp+(1-t)(p+m))dt.$$

Two times partial derivative calculations lead to

$$\frac{\partial^2\ln\mathcal{H}_f(p,p+m)}{\partial p^2}=\int_0^1 T'''(tp+(1-t)(p+m))dt. \tag{3.11}$$

Substituting $v=tp+(1-t)(p+m)$ in the integral of right hand side of (3.11) and noting $T'''(t)$ is odd, and using Lemma 2.6 yield

$$\frac{\partial^2\ln\mathcal{H}_f(p,p+m)}{\partial p^2}=m^{-1}\int_p^{p+m}T'''(v)dv,$$

and then

$$\frac{\partial^2\ln R_f(p,p+m)}{\partial p^2}=m^{-1}\int_{|p|}^{|p+m|}\left(T'''(v;a,b)-T'''(v;c,d)\right)dv. \tag{3.12}$$

Since $\mathcal{T}_3(1,u)$ strictly increase (decrease) with $u>1$ and decrease (increase) with $0<u<1$, for $a,b,c,d>0$ with $b/a>d/c\geq 1$ (3.6) holds. Therefore, the integrand of right hand side of (3.12) is positive (negative) if $|p+m|>|p|$ and negative (positive) if $|p+m|<|p|$, i.e.,

$$\frac{\partial^2\ln R_f(p,p+m)}{\partial p^2}\begin{cases}>(<)0 & \text{if } |p+m|>|p|,\text{ i.e., } p>-m/2,\\ <(>)0 & \text{if } |p+m|<|p|,\text{ i.e., } p<-m/2.\end{cases}$$

This completes the proof. □

Theorem 3.5 (third log-convexity)**.** *Suppose that $f\colon \mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is a symmetric, homogenous and three-time differentiable function and $\mathcal{T}_3(1,u)$ strictly increase (decrease) with $u>1$ and decrease (increase) with $0<u<1$. Then for any $a,b,c,d>0$ with $b/a>d/c\geq 1$ and fixed $m>0$, $R_f(p,2m-p)$ is strictly log-convex (log-concave) in p on $(0,2m)$.*

Proof. By (2.12), $\ln \mathcal{H}_f(p, 2m-p)$ can be expressed in integral form as

$$\ln \mathcal{H}_f(p, 2m-p) = \int_0^1 T'(t_1)dt, \text{ where } t_1 = tp + (1-t)(2m-p),\ p \in (0, 2m),$$

Two times partial derivative calculations lead to

$$\frac{\partial^2 \ln \mathcal{H}_f(p, 2m-p)}{\partial p^2} = \int_0^1 (2t-1)^2 T'''(t_1)dt.$$

Hence,

$$\begin{aligned}\frac{\partial^2 \ln R_f(p, 2m-p)}{\partial p^2} &= \frac{\partial^2 \ln \mathcal{H}_f(p, 2m-p; a, b)}{\partial p^2} - \frac{\partial^2 \ln \mathcal{H}_f(p, 2m-p; c, d)}{\partial p^2} \\ &= \int_0^1 (2t-1)^2 \left(T'''(t_1; a, b) - T'''(t_1; c, d)\right) dt.\end{aligned}$$

Obviously, $t_1 = tp + (1-t)(2m-p) > 0$ due to $0 < p < 2m$ and $0 \le t \le 1$. (3.6) yields $T'''(t_1; a, b) - T'''(t_1; c, d) > (<) 0$, it follows that

$$\frac{\partial^2 \ln R_f(p, 2m-p)}{\partial p^2} > (<) 0 \text{ for } p \in (0, 2m),$$

The proof is accomplished. □

Theorem 3.6 (fourth log-convexity)**.** *Suppose that $f \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a symmetric, homogenous and three-time differentiable function and $\mathcal{T}_3(1, u)$ strictly increase (decrease) with $u > 1$ and decrease (increase) with $0 < u < 1$. Then for any $a, b, c, d > 0$ with $b/a > d/c \ge 1$ and fixed $r, s \in \mathbb{R}$ with $r + s \ne 0$, $R_f(pr, ps)$ is strictly log-convex (log-concave) in p on $(0, \infty)$ and log-concave (log-convex) on $(-\infty, 0)$ if $r + s > 0$, and strictly log-concave (log-convex) in p on $(0, \infty)$ and log-convex (log-concave) on $(-\infty, 0)$ if $r + s < 0$.*

Proof. Taking the logarithm of $\mathcal{H}_f(pr, ps)$ and expressing in integral form yield

$$\ln \mathcal{H}_f(pr, ps) = \begin{cases} \dfrac{1}{r-s} \int_s^r T'(pt)dt & \text{if } r \ne s, \\ T'(pr) & \text{if } r = s. \end{cases} \tag{3.13}$$

We distinguish two cases.

1) Case 1: $r = s$. By Theorem 3.1, we see that function $R_f(pr, pr)$ is strictly log-convex (log-concave) in p if $pr > 0$ and log-concave (log-convex) if $pr < 0$, i.e., $R_f(pr, pr)$ is strictly log-convex (log-concave) in p if $r > 0$ and log-concave (log-convex) if $r < 0$.

2) Case 2: $r \ne s$. Two times partial derivation calculations for (3.13) yield

$$\frac{\partial^2 \ln \mathcal{H}_f(pr, ps)}{\partial p^2} = \frac{1}{r-s} \int_s^r t^2 T'''(pt)dt, \tag{3.14}$$

and then

$$\frac{\partial^2 \ln R_f(pr, ps)}{\partial p^2} = \frac{1}{r-s} \int_s^r t^2 \left(T'''(pt; a, b) - T'''(pt; c, d)\right) dt. \tag{3.15}$$

Note $T'''(t)$ is odd and so is $t^2 T'''(pt)$, and make use of Lemma 2.6, (3.15) can be written as

$$\begin{aligned}\frac{\partial^2 \ln R_f(pr,ps)}{\partial p^2} &= \frac{1}{r-s}\int_{|s|}^{|r|} t^2\left(T'''(pt;a,b)-T'''(pt;c,d)\right)dt\\ &= \frac{r+s}{|r|-|s|}\int_{|s|}^{|r|} t^2\left(T'''(pt;a,b)-T'''(pt;c,d)\right).\end{aligned}$$

From (3.6) it follows that

$$\frac{\partial^2 \ln R_f(pr,ps)}{\partial p^2}\begin{cases} >(<)0 \text{ if } p>0, r+s>0,\\ <(>)0 \text{ if } p<0, r+s>0,\\ <(>)0 \text{ if } p>0, r+s<0,\\ >(<)0 \text{ if } p<0, r+s<0.\end{cases}$$

This implies our desired result is valid.

This proof ends. □

We will close this section by giving the Hermite-Hadamard inequality.

Theorem 3.7. *Suppose that $f\colon \mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is a symmetric, homogenous and three-time differentiable function and $\mathcal{T}_3(1,u)$ strictly increase (decrease) with $u>1$ and decrease (increase) with $0<u<1$. Then for any $a,b,c,d>0$ with $b/a>d/c\geq 1$, the following inequalities*

$$\frac{\mathcal{H}_f(\frac{p+q}{2},\frac{p+q}{2};a,b)}{\mathcal{H}_f(\frac{p+q}{2},\frac{p+q}{2};c,d)} < (>) \frac{\mathcal{H}_f(p,q;a,b)}{\mathcal{H}_f(p,q;c,d)} < (>) \sqrt{\frac{\mathcal{H}_f(p,p;a,b)}{\mathcal{H}_f(p,p;c,d)}\frac{\mathcal{H}_f(q,q;a,b)}{\mathcal{H}_f(q,q;c,d)}} \tag{3.16}$$

hold if $p+q>(<)0$ with $p\neq q$. (3.16) is reversed if $p+q<0$ with $p\neq q$.

Proof. We first verify that

$$\ln\frac{\mathcal{H}_f(t,t;a,b)}{\mathcal{H}_f(t,t;c,d)}+\ln\frac{\mathcal{H}_f(-t,-t;a,b)}{\mathcal{H}_f(-t,-t;c,d)} = 2\ln\frac{\mathcal{H}_f(0,0;a,b)}{\mathcal{H}_f(0,0;c,d)}.$$

In fact, (2.9) and (2.6) yield

$$\begin{aligned}\ln\frac{\mathcal{H}_f(t,t;a,b)}{\mathcal{H}_f(t,t;c,d)}+\ln\frac{\mathcal{H}_f(-t,-t;a,b)}{\mathcal{H}_f(-t,-t;c,d)} &= T'(t;a,b)-T'(t;c,d)+T'(-t;a,b)-T'(-t;c,d)\\ &= 2\left(T'(0;a,b)-T'(0;c,d)\right) = 2\ln\frac{\mathcal{H}_f(0,0;a,b)}{\mathcal{H}_f(0,0;c,d)}.\end{aligned}$$

This indicates that function $t\to\ln R_f(t,t;a,b;c,d)$ $(=\ln\frac{\mathcal{H}_f(t,t;a,b)}{\mathcal{H}_f(t,t;c,d)})$ is symmetric with respect to element $m=0$.

On the other hand, Theorem 3.1 shows that $t\to\ln R_f(t,t;a,b;c,d)$ $(=\ln\frac{\mathcal{H}_f(t,t;a,b)}{\mathcal{H}_f(t,t;c,d)})$ is strictly convex (concave) in t on $(0,+\infty)$ and concave (convex) on $(-\infty,0)$ under the conditions of this theorem.

By Lemma 2.7, for $p+q>0$ with $p\neq q$ we have

$$\ln\frac{\mathcal{H}_f(\frac{p+q}{2},\frac{p+q}{2};a,b)}{\mathcal{H}_f(\frac{p+q}{2},\frac{p+q}{2};c,d)} < (>) \frac{1}{p-q}\int_q^p \ln\frac{\mathcal{H}_f(t,t;a,b)}{\mathcal{H}_f(t,t;c,d)}dt < (>) \frac{\ln\frac{\mathcal{H}_f(p,p;a,b)}{\mathcal{H}_f(p,p;c,d)}+\ln\frac{\mathcal{H}_f(q,q;a,b)}{\mathcal{H}_f(q,q;c,d)}}{2}, \tag{3.17}$$

which, combined with (2.12), is equivalent to (3.16). Similarly, if $p+q<0$ with $p\neq q$ then (3.16) is reversed.

This proof is finished. □

Remark 3.8. *It is clear that Theorem 3.7 generalizes and improves main result given in* [24, Corollary 2].

4. An Application for Stolarsky Means

As an application for Stolarsky means, we have the following.

Theorem 4.1. *Let $a, b, c, d > 0$ with $b/a > d/c \geq 1$. Then*

1) for fixed $q \in \mathbb{R}$, the ratio of Stolarsky means $R_{p,q}(a, b; c, d) = \frac{S_{p,q}(a,b)}{S_{p,q}(c,d)}$ is strictly log-concave in p on $(\frac{|q|-q}{2}, \infty)$ and log-convex on $(-\infty, -\frac{|q|+q}{2})$;

2) for fixed $m \geq 0$, the ratio of Stolarsky means $R_{p,p+m}(a, b; c, d) = \frac{S_{p,p+m}(a,b)}{S_{p,p+m}(c,d)}$ is strictly log-concave in p on $(-m/2, \infty)$ and log-convex on $(-\infty, -m/2)$;

3) for fixed $m > 0$, the ratio of Stolarsky means $R_{p,2m-p}(a, b; c, d) = \frac{S_{p,2m-p}(a,b)}{S_{p,2m-p}(c,d)}$ is strictly log-concave in p on $(0, 2m)$;

4) for fixed $r, s \in \mathbb{R}$, the ratio of Stolarsky means $R_{pr,ps}(a, b; c, d) = \frac{S_{pr,ps}(a,b)}{S_{pr,ps}(c,d)}$ is strictly log-concave in p on $(0, \infty)$ and log-convex on $(-\infty, 0)$ if $r + s > 0$, and strictly log-convex on $(0, \infty)$ and log-concave on $(-\infty, 0)$ if $r + s < 0$.

Proof. In [23] we have shown $S_{p,q}(a, b) = \mathcal{H}_L(p, q; a, b)$, where

$$L = L(x, y) = \frac{x-y}{\ln x - \ln y}(x, y > 0, x \neq y), L(x, x) = x.$$

Obviously, logarithmic mean $(x, y) \to L(x, y)$ is a positive, symmetric, homogenous and three-time differentiable function. Thus, it suffices to check the monotonicity of $\mathcal{T}_3(1, u)$.

Some simple calculations result in

$$\begin{aligned}
\mathcal{I} &= (\ln L)_{xy} = \frac{1}{(x-y)^2} - \frac{1}{xy(\ln x - \ln y)^2}, \\
(x\mathcal{I})_x &= -\frac{x+y}{(x-y)^3} + \frac{2}{xy(\ln x - \ln y)^3}, \\
\mathcal{T}_3(x, y) &= -xy(x\mathcal{I})_x \ln^3(x/y) = -2 + \frac{xy(x+y)}{(x-y)^3}\ln^3(x/y), \\
\frac{d\mathcal{T}_3(1, u)}{du} &= 6u(u-1)^4 \ln^3 u \quad \left. \frac{u^2-1}{\ln u^2} - \frac{\frac{u^2+1}{2} + 2\sqrt{u^2}}{3}\right).
\end{aligned}$$

Making use of the well-known inequality $L(x, y) < \dfrac{\frac{x+y}{2} + 2\sqrt{xy}}{3}$ ($x, y > 0$ with $x \neq y$) given in [2], we see that $\frac{d\mathcal{T}_3(1,u)}{du} < 0$ if $u > 1$ and $\frac{d\mathcal{T}_3(1,u)}{du} > 0$ if $0 < u < 1$.

Applying Theorem 3.2, 3.5 and 3.6, we obtain the first, third and fourth results; applying Theorem 3.1 ($m = 0$) and 3.4 ($m > 0$) yield the second one.

This completes the proof. □

Putting $d = c$ in the first result of Theorem 4.1, we have the following result.

Corollary 4.2. *For fixed $q \in \mathbb{R}$, Stolarsky means $S_{p,q}(a, b)$ is strictly log-concave in p on $(\frac{|q|-q}{2}, \infty)$ and log-convex on $(-\infty, -\frac{|q|+q}{2})$.*

Hence we can see that the first result of Theorem 4.1 generalizes and improves F. Qi's given in [16].

Remark 4.3. *Putting $d = c$ in the second result of Theorem 4.1, we obtain a result given by W. -S. Cheung and F. Qi in* [8].

Theorem 4.4. *Let $a, b, c, d > 0$ with $b/a > d/c \geq 1$. Then for $p+q > 0$ with $p \neq q$ the following inequalities*

$$\sqrt{\frac{I_p(a,b)}{I_p(c,d)} \frac{I_q(a,b)}{I_q(c,d)}} < \frac{S_{p,q}(a,b)}{S_{p,q}(c,d)} < \frac{I_{\frac{p+q}{2}}(a,b)}{I_{\frac{p+q}{2}}(c,d)} \tag{4.1}$$

hold, where $I_p(a,b) = I^{1/p}(a^p, b^p)$ if $p \neq 0$ and $I_0(a,b) = G(a,b) = \sqrt{ab}$, $I(a,b) = e^{-1}(a^a/b^b)^{1/(a-b)}$ if $a \neq b$ and $I(a,a) = a$. (4.1) is reversed if $p+q < 0$ with $p \neq q$.

Proof. The proof of Theorem 4.1 shows that all conditions of Theorem 3.7 are satisfied. Note $\mathcal{H}_L(p,p;a,b) = I^{1/p}(a^p, b^p)$ if $p \neq 0$ and $I_0(a,b) = G(a,b) = \sqrt{ab}$, $I(a,b) = e^{-1}(a^a/b^b)^{1/(a-b)}$ if $a \neq b$ and $I(a,a) = a$ is the identric (exponential) mean, according to Theorem 3.7 our result required is derived immediately. □

Remark 4.5. *Putting $d = c$ in Theorem 4.4 yields a result presented by P. Czinder and Z. Páles in* [9].

Applying the log-convexity of ratio of Stolarsky means, we can deduce many inequalities of ratio of classical bivariate means. For this end, let us recall the following notatoins.

$$\begin{aligned}
S_{1,0}(a,b) &= L(a,b) \text{ –logarithmic mean,} \\
S_{1,1}(a,b) &= I(a,b) \text{ –identric (exponential) mean,} \\
S_{2,1}(a,b) &= A(a,b) \text{ –arithmetic mean,} \\
S_{3/2,1/2}(a,b) &= He(a,b) \text{ –Hornian mean.} \\
S_{2p,p}(a,b) &= A_p(a,b) = A^{1/p}(a^p, b^p) \text{ –}p\text{-order power mean,} \\
S_{3p/2,p/2}(a,b) &= He_p(a,b) = He^{1/p}(a^p, b^p)\text{–}p\text{-order Horniran mean.}
\end{aligned}$$

we illustrate below.

Corollary 4.6. *Let $a, b, c, d > 0$ with $b/a > d/c \geq 1$. Then inequalities of ratio of means*

$$\frac{L(a,b)}{L(c,d)} < \frac{He_{1/2}(a,b)}{He_{1/2}(c,d)} < \frac{A_{1/3}(a,b)}{A_{1/3}(c,d)} < \frac{I_{1/2}(a,b)}{I_{1/2}(c,d)} \tag{4.2}$$

hold.

Proof. Using first result of Theorem 4.1, we have

$$\left(\frac{S_{1,1}(a,b)}{S_{1,1}(c,d)}\right)^{2/3} \left(\frac{S_{1,1/4}(a,b)}{S_{1,1/4}(c,d)}\right)^{1/3} < \frac{S_{1,3/4}(a,b)}{S_{1,3/4}(c,d)}.$$

It is equivalent to

$$\frac{S_{1,1}(a,b)}{S_{1,1}(c,d)} < \left(\frac{S_{1,3/4}(a,b)}{S_{1,3/4}(c,d)}\right)^{3/2} \left(\frac{S_{1,1/4}(a,b)}{S_{1,1/4}(c,d)}\right)^{-1/2} = \frac{S_{3/4,1/4}(a,b)}{S_{3/4,1/4}(c,d)},$$

which is the first inequality of (4.2).

In the same way, we have

$$\left(\frac{S_{1/3,3/4}(a,b)}{S_{1/3,3/4}(c,d)}\right)^{5/6}\left(\frac{S_{1/3,1/4}(a,b)}{S_{1/3,1/4}(c,d)}\right)^{1/6}<\frac{S_{1/3,2/3}(a,b)}{S_{1/3,2/3}(c,d)}.$$

Note $S_{1/3,3/4}^{5/6}(a,b)S_{1/3,1/4}^{1/6}(a,b)=S_{1/4,3/4}(a,b)=He_{1/2}(a,b)$, second one of (4.2) follows.

From

$$\sqrt{\frac{S_{2/3,1/2}(a,b)}{S_{2/3,1/2}(c,d)}\frac{S_{1/3,1/2}(a,b)}{S_{1/3,1/2}(c,d)}}<\frac{S_{1/2,1/2}(a,b)}{S_{1/2,1/2}(c,d)}$$

and note $\sqrt{S_{2/3,1/2}(a,b)S_{1/3,1/2}(a,b)}=S_{1/3,2/3}(a,b)=A_{1/3}(a,b)$, third one of (4.2) follows.

This completes the proof. □

Remark 4.7. *Putting $d=c$ in Corollary 4.6 yields Jia-Cao inequality* [11], *Lin inequality* [12] *and Stolarsky inequality* [20].

Corollary 4.8. *Let $a,b,c,d>0$ with $b/a>d/c\geq 1$. Then the following inequalities of ratio of means*

$$\frac{L(a,b)}{L(c,d)} < \left(\frac{He_{1/2}(a,b)}{He_{1/2}(c,d)}\right)^{4}\left(\frac{A_{1/3}(a,b)}{A_{1/3}(c,d)}\right)^{-3}, \tag{4.3}$$

$$\frac{I_{1/2}(a,b)}{I_{1/2}(c,d)} < \left(\frac{He_{1/2}(a,b)}{He_{1/2}(c,d)}\right)^{-2}\left(\frac{A_{1/3}(a,b)}{A_{1/3}(c,d)}\right)^{3} \tag{4.4}$$

hold.

Proof. By third result of Theorem 4.1, the ratio of Stolarsky means $R_{p,1-p}(a,b;c,d)=\frac{S_{p,1-p}(a,b)}{S_{p,1-p}(c,d)}$ is strictly log-concave in p on $[0,1]$. Making use of property of log-concave functions gives

$$\left(\frac{S_{1,0}(a,b)}{S_{1,0}(c,d)}\right)^{1/4}\left(\frac{S_{2/3,1/3}(a,b)}{S_{2/3,1/3}(c,d)}\right)^{3/4}<\frac{S_{3/4,1/4}(a,b)}{S_{3/4,1/4}(c,d)},$$

that is

$$\left(\frac{L(a,b)}{L(c,d)}\right)^{1/4}\left(\frac{A_{1/3}(a,b)}{A_{1/3}(c,d)}\right)^{3/4}<\frac{He_{1/2}(a,b)}{He_{1/2}(c,d)},$$

which is equivalent to (4.3).

Similarly, from

$$\left(\frac{S_{1/2,1/2}(a,b)}{S_{1/2,1/2}(c,d)}\right)^{1/3}\left(\frac{S_{3/4,1/4}(a,b)}{S_{3/4,1/4}(c,d)}\right)^{2/3}<\frac{S_{2/3,1/3}(a,b)}{S_{2/3,1/3}(c,d)}$$

it follows (4.4).

This proof ends. □

Remark 4.9. *Putting $d=c$ in Corollary 4.8 yields*

$$L(a,b) < He_{1/2}^{4}(a,b)A_{1/3}^{-3}(a,b), \tag{4.5}$$

$$I_{1/2}(a,b) < He_{1/2}^{-2}(a,b)A_{1/3}^{3}(a,b). \tag{4.6}$$

Since $L<He_{1/2}(a,b)<A_{1/3}(a,b)$ inequality (4.5) is stronger than Jia-Cao inequality [11] *and Lin inequality* [12]. *While inequality (4.6) clearly is a newcomer.*

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