DE GRUYTER

Journal of Economy and its Applications
ISSN: 2217-7973, URL: http://ilirias.com/jea
http://www.degruyter.com/view/j/jea
DOI: 10.1515/jea-2016-0001
Volume 6 Issue 1(2016), Pages 1-27.

Numerical Analysis of Equilibrium Determinacy in the New Keynesian Model with Monetary and Fiscal Policy Lags

Eiji Tsuzuki*, Shunsuke Shinagawa†, and Tomohiro Inoue‡

Abstract

This study provides a theoretical analysis on the effects of monetary and fiscal policy lags on equilibrium determinacy. Local equilibrium determinacy is shown to be achieved by applying active monetary and passive fiscal policy in discrete-time New Keynesian (NK) models that include a fiscal policy rule with a time lag in policy response. However, in models with money-in-the-production function formation, equilibrium indeterminacy can occur even under these policy actions. We show that the above-mentioned policy implications can be derived from a continuous-time NK model that does not introduce a policy lag. We then introduce monetary and fiscal policy lags into the model and demonstrate that both or either of these policy lags can resolve the problem of indeterminacy.

JEL Classification: E32; E52

Keywords: New Keynesian model, policy lag, two-delay differential equations, determinacy analysis

*Corresponding author. Faculty of Economics, Nanzan University, Nagoya 466-8673, Japan; Tel: +81-52-832-3111(ext.3821); E-mail: tsuzuki@nanzan-u.ac.jp

†Faculty of Economics, Kanagawa University, Yokohama 221-8686, Japan; E-mail: shinagawa@kanagawa-u.ac.jp

‡Faculty of Economics, Komazawa University, Tokyo 154-0012, Japan; E-mail: inoue-tomo@gmail.com

1 Introduction

This study examines the effects of monetary and fiscal policy lags on local equilibrium determinacy by developing a continuous-time New Keynesian (NK) model. The model economic system is expressed as delay-differential equations. Initially, few NK scholars examined such a type of system. Moreover, policymakers may face a situation of delay in policy implementation due to political reasons. Hence, we believe that this study is important from both theoretical and practical aspects.

NK models have often been used as framework to analyze the effects of monetary policies on equilibrium determinacy.[1] These models are based on the optimizing behavior of economic agents. In addition, the models assume the stickiness associated with changes in prices and nominal wages, implying that they are short-term or medium-term models.

In the simplest NK model, wherein the nominal interest rate responds only to the inflation rate, the following policy norm can be obtained: to achieve local equilibrium determinacy, the monetary policy should be "active"; that is, the nominal interest rate's response should be more than one unit in case of a one-unit change in inflation rate. (Conversely, if the nominal interest rate's response is less than one unit, the policy is referred to as "passive.") This is the well-known proposition known as the "Taylor principle."

Some notable studies have demonstrated that even if the Taylor principle is not satisfied, equilibrium determinacy can be established. For example, Meng and Yip (2004), Bilbiie (2008), and Gliksberg (2009) show, respectively, that equilibrium determinacy can be achieved by assuming endogenous investment, limited asset market participation, and the existence of capital adjustment costs.

Furthermore, Leeper (1991) examined the interaction between monetary and fiscal policies by using an optimizing model that does not include price and nominal wage stickiness. As for fiscal policy, he assumed that the fiscal authority changes the amount of lump-sum tax according to fluctuations in total government liabilities. If the fiscal authority implements a policy without abiding by budget discipline, the policy is considered active. Conversely, if the fiscal authority seeks to keep the total government liabilities consistent with income, implying a Ricardian-type fiscal regime, the policy is considered passive. Leeper (1991) gives two possible combinations of monetary and fiscal policies for local equilibrium determinacy: an active monetary policy should be combined with a passive fiscal policy, and a passive monetary policy should be combined with an active fiscal policy.

Furthermore, Schmitt-Grohé and Uribe (2007) and Kumhof et al. (2010) developed Leeper's model to include price stickiness; that is, they developed Leeper's model into NK models. Their analyses followed Leeper (1991) by assuming rules based on the responses of not only lump-sum taxes but also the income tax rate to the total government liabilities. Their studies basically confirmed Leeper's results.

In these studies, tax rates are assumed to respond to a past (one period earlier) value of government liabilities. In other words, they assume the presence of a time lag. This situation can be described by using general terms as follows: a variable evaluated at period t, z_t, responds to another variable evaluated not at time t, x_t, but at time $t-1$, x_{t-1}. In discrete-time models, as those referred to above, the characterization of the systems' dynamics involves no analytical difficulty, because they are simply standard systems of difference equations. However, in continuous-time models, the presence of a time lag considerably complicates the analysis because the models include terms $\dot{x}_t$, x_t, and $x_{t-\theta}$, where θ represents a time lag. This is the well-known system called "delay-differential equation system" (in other words, differential-difference equation system).[2]

Tsuzuki (2014, 2015) developed continuous-time NK models that introduce a time lag into monetary policy implementation.[3] He demonstrates that a time lag may resolve the problem of indeterminacy in models with money-in-the-production-function (MIPF). In the present study, we develop a similar model that includes both monetary and fiscal policy lags and perform a local equilibrium determinacy analysis. For the analysis, we use a numerical method developed by Lin and Wang (2012) in order to visualize the interaction between the two policy lags.

This paper is structured as follows. In Section 2, we discuss the behavior of economic agents in a model economy. Section 3 analyzes the local dynamics of a case where there is no time lag. Section 4 evaluates a case indicating monetary and fiscal policy lags. Section 5 concludes the paper.

2 The model

In this section, we propose a simple continuous-time NK model following Benhabib et al. (2003). We construct the model economy using monetary and fiscal authorities and household–firm units indexed by i (we normalize their total at unity; that is, $i \in [0, 1]$). Household–firm unit i produces and sells good i under monopolistic competition. Furthermore, household–firm units aggregate the heterogeneous types of goods and then consume them as a composite good.

We describe monetary and fiscal policy implementations following Leeper (1991) and Schmitt-Grohé and Uribe (2007). Monetary authorities manipulate the nominal interest rate according to fluctuations in the inflation rate, whereas the fiscal authority manipulates the income tax rate according to fluctuations in total government liabilities (i.e., money and bonds). Leeper (1991) examines the case of lump-sum tax, a non-distortionary tax, and Schmitt-Grohé and Uribe (2007) examine the case of income tax, a distortionary tax. This study deals with the latter, income tax, emphasizing the generality of specification. For simplicity, we assume that the fiscal authority spends its revenue but its expenditure does not affect the production or utility of household–firm units.

First, we describe the demand for heterogeneous goods and the aggregation of such goods by household–firm units.

2.1 Intratemporal optimization

Each household–firm unit aggregates various types of goods via the Dixit–Stiglitz function[4] :

$$y = \left[\int_0^1 y_i^{\frac{\phi-1}{\phi}} di \right]^{\frac{\phi}{\phi-1}}, \tag{1}$$

where y is the quantity of the composite good, y_i is the quantity of good i, and ϕ (> 1) is the elasticity of substitution between heterogeneous goods.[5]

Given the quantity of a composite good, the price of composite good p, and the price of good i, p_i, the demand for good i is determined by minimizing the cost $\int_0^1 p_i y_i di$ subject to Equation (1). The first-order optimality conditions for this problem yield the following expression[6] :

$$y_i = \left(\frac{p_i}{p}\right)^{-\phi} y, \tag{2}$$

where $p = \left[\int_0^1 p_i^{1-\phi} di\right]^{\frac{1}{1-\phi}}$. Equation (2) is a demand function for good i.

2.2 Intertemporal optimization

In this subsection, we describe the consumption behavior, price setting, and production performance of household–firm units. Household–firm unit i produces good i using labor force ℓ_i. Assuming a linear technology, we specify the production function as follows:

$$y_i = z\ell_i, \tag{3}$$

where z is a constant representing productivity.

Household–firm units obtain utility from consumption and money holdings and disutility from labor supply and price revisions. We specify the utility function as follows:

$$U(c_i, m_i, \ell_i, \pi_i) = u(c_i, m_i) - \frac{\ell_i^{1+\psi}}{1+\psi} - \frac{\eta}{2}(\pi_i - \pi^*)^2, \tag{4}$$

where c_i is the consumption of a composite good, m_i is real money balance, $\pi_i = \dot{p}_i/p_i$ is the price change rate of good i (π^* denotes the steady-state value of π), $\psi > 0$, and $\eta > 0$.[7] We use the following normal assumptions for utility function: $u_c = \partial u/\partial c_i > 0$, $u_{cc} = \partial^2 u/\partial c_i^2 < 0$, $u_m = \partial u/\partial m_i > 0$, and $u_{mm} = \partial^2 u/\partial m_i^2 < 0$. Because of the existence of the price revision cost, $\frac{\eta}{2}(\pi_i - \pi^*)^2$, prices become sticky. Thus, η can be interpreted as representing price stickiness.

The assets of household–firm unit i consist of money and bonds: $A_i = M_i + B_i$, where A_i represents nominal assets, M_i represents nominal money balances, and B_i represents nominal bonds. Assets can be increased through income and bond interest inflows and decreased through consumption and income tax outflows. Thus, the following equation holds: $\dot{A}_i = (1-\tau)p_i y_i + RB_i - pc_i$, where R is the nominal interest rate for bonds and τ is the income tax rate. We can rewrite this equation in real terms as follows:

$$\dot{a}_i = (1-\tau)\frac{p_i}{p}y_i + ra_i - c_i - Rm_i, \tag{5}$$

where $a_i = A_i/p$ represents real assets and $r = R - \pi$ ($\pi = \dot{p}/p$) is the real interest rate.

Household–firm unit i determines c_i, m_i, and π_i by maximizing its discounted utility stream represented by $\int_0^\infty U(c_i, m_i, \ell_i, \pi_i)e^{-\rho t}dt$ (where $\rho > 0$ is the discount rate), subject to the restrictions of Equations (2), (3), (5), and $\pi = \dot{p}/p$.

The current-value Hamiltonian function for this optimization problem can be expressed as follows:

$$\begin{aligned}\mathcal{H} = u(c_i, m_i) - \frac{1}{1+\psi}\left(\frac{1}{z}\left(\frac{p_i}{p}\right)^{-\phi} y\right)^{1+\psi} - \frac{\eta}{2}(\pi_i - \pi^*)^2 \\ + \mu_1\left[(1-\tau)\frac{p_i}{p}\left(\frac{p_i}{p}\right)^{-\phi} y + ra_i - c_i - Rm_i\right] + \mu_2 \pi_i p_i,\end{aligned}$$

where μ_1 and μ_2 are the co-state variables of state variables a_i and p_i, respectively. The first-order conditions for optimality are

$$\frac{\partial \mathcal{H}}{\partial c_i} = u_c(c_i, m_i) - \mu_1 = 0, \tag{6}$$

$$\frac{\partial \mathcal{H}}{\partial m_i} = u_m(c_i, m_i) - \mu_1 R = 0, \tag{7}$$

$$\frac{\partial \mathcal{H}}{\partial \pi_i} = -\eta(\pi_i - \pi^*) + \mu_2 p_i = 0, \tag{8}$$

$$\dot{\mu}_1 = \rho\mu_1 - \frac{\partial \mathcal{H}}{\partial a_i} = \rho\mu_1 - r\mu_1, \tag{9}$$

$$\dot{\mu}_2 = \rho\mu_2 - \frac{\partial \mathcal{H}}{\partial p_i} = \rho\mu_2 - \left(\frac{y_i}{z}\right)^{\psi}\frac{\phi}{z}\frac{y_i}{p_i} - \mu_1(1-\phi)(1-\tau)\frac{y_i}{p} - \mu_2\pi_i. \tag{10}$$

The second-order conditions are expressed as follows:

$$u_{cc} < 0;\ D \equiv u_{cc}u_{mm} - u_{cm}^2 > 0. \tag{11}$$

Furthermore, economically significant solutions would require satisfying the transversality conditions expressed as

$$\lim_{t\to\infty} a_i(t)e^{-\rho t} = 0,$$
$$\lim_{t\to\infty} p_i(t)e^{-\rho t} = 0.$$

As the behavior of all the household–firm units is based on the same equations (i.e., they are symmetric), we can drop subscript i from all variables. Thus, from Equations (8) and (10), we obtain the following equation:

$$\dot{\pi} = \rho(\pi - \pi^*) - \frac{\phi}{\eta} z^{-(1+\psi)} y^{1+\psi} + \frac{(\phi-1)(1-\tau)}{\eta}\mu_1 y. \tag{12}$$

This is referred to as the NK Phillips curve.

Furthermore, by solving Equation (7) for m, we obtain

$$m = m(c, \mu_1, R); \tag{13}$$
$$m_c = \frac{\partial m}{\partial c} = -\frac{u_{cm}}{u_{mm}};\ u_\mu = \frac{\partial m}{\partial \mu_1} = \frac{R}{u_{mm}} < 0;\ m_R = \frac{\partial m}{\partial R} = \frac{\mu_1}{u_{mm}} < 0.$$

By substituting this expression into Equation (6) and solving for c, we obtain

$$c = c(\mu_1, R); \tag{14}$$
$$c_\mu = \frac{\partial c}{\partial \mu_1} = \frac{u_{mm} - u_{cm}R}{D};\ c_R = \frac{\partial c}{\partial R} = -\frac{u_{cm}\mu_1}{D}.$$

2.3 Monetary policy

2.3.1 Interest rate rule

A monetary policy rule reflecting the behavior of monetary authorities and where the nominal interest rate is changed in response to fluctuations in the inflation rate can be expressed as follows:

$$R = R(\pi);\ R'(\pi) > 0;\ R(\pi^*) = \bar{R}, \tag{15}$$

where $\bar{R}$ is the nominal interest rate corresponding to the target inflation rate, which is considered to be its steady-state value here; $R'(\pi^*) > 1$ represents an active monetary policy, and $R'(\pi^*) < 1$ represents a passive one. This type of monetary policy rule is adopted in Leeper (1991) and is the most popular specification in NK economic theory. Indeed, inflation targeting policies are observed in many developed countries. In some models, the output and asset prices are added as target variables; however, we investigate the simplest and most standard version.

2.3.2 Generalized interest rate rule

The generalized interest rate rule, which postulates that the nominal interest rate responds to a weighted stream of inflation rates, can be represented as follows:

$$R(t) = R(\pi^g(t));\ \pi^g(t) \equiv \int_{-\infty}^{t} \delta(s)\pi(s)ds, \tag{16}$$

where $\delta(s)$ is a weighting factor for the inflation rate stream, $\int_{-\infty}^{t} \pi(s)ds$, and is defined as follows:

$$\delta(s) = \left(\frac{n}{\theta_1}\right)^n \frac{(t-s)^{n-1}}{(n-1)!} e^{-\frac{n}{\theta_1}(t-s)},$$

where n takes positive integer numbers, $\theta_1 > 0$, and $\int_{-\infty}^{t} \delta(s)ds = 1$. The mean of this function is given by θ_1 and the variance is given by θ_1^2/n.

If $n \to 1$, then $\delta(s)$ becomes an exponential function, $(1/\theta_1)e^{-(1/\theta_1)(t-s)}$, implying that monetary authorities place greatest emphasis on the present (see Fig. 1).[8] The backward-looking interest rate rule examined by Benhabib et al. (2003) corresponds to this case, wherein θ_1 measures the degree to which the monetary authority is backward looking. When $n \geq 2$, $\delta(s)$ becomes a unimodal function that becomes maximum at $s = t - (n-1)\theta_1/n$, and when $n \to \infty$, it becomes a vertical line at $t - \theta_1$.

[Figure 1]

Therefore, the standard monetary policy rule in Equation (15) can be considered a special case of Equation (16) corresponding to the case where $n \to \infty$ and $\theta_1 \to 0$.

2.3.3 Interest rate rule with a delay

When a delay is present in an interest rate's response to fluctuations in the inflation rate, the interest rate rule can be expressed as follows:

$$R(t) = R(\pi(t - \theta_1)). \tag{17}$$

This expression corresponds to the case where $n \to \infty$ in Equation (16), and θ_1 represents a time lag in monetary policy implementation.

2.4 Fiscal policy

The budget constraint equation for the public sector is expressed as follows: $\dot{B} = RB - \dot{M} - \tau py + pg$, where g represents the real government spending, which is assumed to be constant. By rewriting this equation in real terms, we obtain

$$\dot{a} = ra - Rm - \tau y + g. \tag{18}$$

2.4.1 Tax rate rule

As adopted in Schmitt-Grohé and Uribe (2007), manipulation of the income tax rate according to fluctuations in the total real government liabilities, a, by the fiscal authority can be expressed as follows:

$$\tau = \tau(a);\ \tau'(a) > 0;\ \tau(a^*) = \bar{\tau}, \tag{19}$$

where $\bar{\tau}$ is the income tax rate corresponding to the target level of total government liabilities, which is considered as its steady-state value. Schmitt-Grohé and Uribe (2007) only considered the case wherein a one period policy delay exists in the government's responses. However, we compare the case with a positive policy lag and the case without a lag to emphasize the effects of policy lags of economic stability.

By substituting Equation (19) into Equation (18) and focusing on the dynamics of a, we find that the dynamic path of the total government liabilities is locally stable if

$$\frac{r^*}{y^*} - \tau'(a^*) < 0.$$

Therefore, following the terminology of Leeper (1991), $\tau'(a^*) > r^*/y^*$ represents a passive fiscal policy and $\tau'(a^*) < r^*/y^*$ represents an active one.

2.4.2 Tax rate rule with a delay

In case of a delay in the fiscal authority's response to fluctuations in total government liabilities, the tax rate rule is rewritten as follows:

$$\tau(t) = \tau(a(t-\theta_2)), \tag{20}$$

where θ_2 represents a time lag in fiscal policy implementation.

3 Case with no policy lags

To emphasize the effects of policy lags on local determinacy, we first consider the local dynamics of the model economic system with no policy lags. By using the equilibrium condition $y = c + g$ for the goods market, the model economy expressed in Equations (9), (12)–(15), and (19) can be summarized in the following equation system:

$$\begin{aligned}
\dot{\mu}_1 &= [\rho - R(\pi) + \pi]\mu_1, \\
\dot{\pi} &= \rho(\pi - \pi^*) - \frac{\phi}{\eta} z^{-(1+\psi)}[c(\mu_1, R(\pi)) + g]^{1+\psi} + (1-\tau(a))\frac{\phi-1}{\eta}\mu_1[c(\mu_1, R(\pi)) + g], \\
\dot{a} &= [R(\pi) - \pi]a - R(\pi)m(c(\mu_1, R(\pi)), \mu_1, R(\pi)) - \tau(a)[c(\mu_1, R(\pi)) + g] + g.
\end{aligned} \tag{21}$$

The steady-state values of System (21) can be expressed as (μ_1^*, π^*, a^*), which satisfies the simultaneous equations as follows:

$$\begin{aligned}
\pi^* &= \bar{R} - \rho, \\
\mu_1^*[c(\mu_1^*, \bar{R}) + g]^{-\psi} &= \frac{\phi}{(1-\bar{\tau})(\phi-1)} z^{-(1+\psi)}, \\
a^* &= \frac{\bar{R}m(c(\mu_1^*, \bar{R}), \mu_1^*, \bar{R}) + \bar{\tau}(c(\mu_1^*, \bar{R}) + g) - g}{\rho}.
\end{aligned} \tag{22}$$

Furthermore, the Jacobian matrix of System (21) evaluated at the steady state can be expressed as

$$J_1 = \begin{bmatrix} 0 & -(R'-1)\mu_1^* & 0 \\ P_1 & \rho - P_2 R' & -\tau' \frac{\phi-1}{\eta}\mu_1^* y^* \\ P_3 & (R'-1)a^* + P_4 R' & r^* - \tau' y^* \end{bmatrix},$$

where[9]

$$P_1 = -\psi\frac{\phi}{\eta}z^{-(1+\psi)}[c(\mu_1^*, \bar{R}) + g]^{\psi}c_\mu + \frac{(1-\bar{\tau})(\phi-1)}{\eta}[c(\mu_1^*, \bar{R}) + g],$$
$$P_2 = \psi\frac{\phi}{\eta}z^{-(1+\psi)}[c(\mu_1^*, \bar{R}) + g]^{\psi}c_R,$$
$$P_3 = -\bar{R}(m_c c_\mu + m_\mu) - \bar{\tau}c_\mu,$$
$$P_4 = -[m(c(\mu_1^*, \bar{R}), \mu_1^*, \bar{R}) + \bar{R}(m_c c_R + m_R) + \bar{\tau}c_R].$$

Equation (14) indicates that the following relationships hold: $P_2 \gtreqless 0 \iff c_R \gtreqless 0 \iff u_{cm} \lesseqgtr 0$. As shown by Feenstra (1986) and Carlstrom and Fuerst (2003), the money-in-the-production-function (MIPF) model can be seen as a special case of money-in-the-utility-function (MIUF) model. In the case of a negative correlation between consumption and real money balances (i.e., $u_{cm} < 0$), the MIUF model is equivalent to the MIPF model. Therefore, the case of $P_2 > 0$ can be considered equivalent to an MIPF model.

The characteristic equation for System (21) can be given by

$$\Delta_1(\lambda) \equiv |\lambda I - J_1| = \lambda^3 + v_1\lambda^2 + v_2\lambda + v_3 = 0, \tag{23}$$

where

$$v_1 = -(\rho - P_2R') - (r^* - \tau' y^*),$$
$$v_2 = P_1(R'-1)\mu_1^* + (\rho - P_2R')(r^* - \tau' y^*) + [(R'-1)a^* + P_4R']\tau'\frac{\phi-1}{\eta}\mu_1^* y^*,$$
$$v_3 = -P_1(R'-1)\mu_1^*(r^* - \tau' y^*) - P_3(R'-1)\mu_1^{*2}\tau'\frac{\phi-1}{\eta}y^*.$$

Equation (23) has three roots. Because c and π are both jump variables and a is a non-jump variable, the equilibrium can only be locally determinate when Equation (23) includes exactly two roots with positive real parts.

3.1 Conditions for determinacy

As the value of $detJ_1$ $(= -v_3)$ equals the product of the roots, $v_3 > 0$ must hold for determinacy. In this case, the signs of the three roots are $(++-)$ or $(---)$. In addition, if at least one of the conditions for Routh–Hurwitz stability[10] (which provides necessary and sufficient conditions for the real parts of all the roots to be negative: $v_1 > 0$; $v_2 > 0$; $v_3 > 0$; and $v_4 \equiv v_1 v_2 - v_3 > 0$) is *not* satisfied, then we can identify the pattern of the signs as $(++-)$. In this case, the equilibrium is locally determinate.

We now focus on R' and τ', which indicate the responsiveness of monetary and fiscal policies, respectively. Assume that

$$\begin{aligned} V_1 &= \{(R', \tau') : v_1 < 0\}, \\ V_2 &= \{(R', \tau') : v_2 < 0\}, \\ V_3 &= \{(R', \tau') : v_3 > 0\}, \\ V_4 &= \{(R', \tau') : v_4 < 0\}. \end{aligned}$$

Then, we can characterize the set of (R', τ') that achieves local determinacy as follows:

$$Determinacy = \{V_1 \cup V_2 \cup V_4\} \cap V_3.$$

Henceforth, we would have to depend on a numerical method.

3.2 Numerical simulations

To perform numerical analysis, we specify the utility function as follows:

$$u(c, m) = \frac{(cm)^{1-\sigma} - 1}{1 - \sigma},$$

where σ is the inverse of the intertemporal elasticity of substitution in consumption. Other structural parameters are set as follows, in conjunction with those outlined in Benhabib et al. (2003) (quarterly values): $\phi = 21$; $\psi = 1$; $\sigma = 2$; $\eta = 350$; $\rho = 0.005$; and $\bar{R} = 0.015$. For $\sigma > 1$, $c_R < 0$ (hence, $P_2 > 0$) holds. Therefore, in this case, our model becomes equivalent to an MIPF model.

Furthermore, the steady-state value of the income tax rate is set to equal its average level of 0.2. In addition, g is set at 0.073 for the share of government expenditure in the national income, g/y^*, to match its realistic value of 0.19. Finally, constant productivity z is set at unity. Under these assumptions, we obtain the illustration of sets V_1–V_4, as shown in Fig. 2.

[Figure 2]

The results implied by Fig. 2 are partly consistent with those of Kumhof et al. (2010); that is, a local equilibrium determinacy can be realized using a passive monetary policy ($R' < 1$) accompanied by an active fiscal policy ($\tau' < r^*/y^* = 0.013$)[11] . However, unlike in Kumhof et al.'s (2010) model, local determinacy cannot be achieved when monetary policy is active ($R' > 1$) and fiscal policy is passive ($\tau' > r^*/y^*$). In this case, the signs of the real parts of all roots are negative. Thus, indeterminacy occurs.

Kumhof et al. (2010) introduce money via a cash-in-advance constraint. This assumption creates a situation similar to an MIUF model that corresponds to the case of $P_2 < 0$ in our model. If we assume $\sigma < 1$, then $P_2 < 0$ holds. For example, by assuming $\sigma = 0.9$ and describing the sets V_1–V_4 on the R'-τ' plane, we obtain Fig. 3. This figure demonstrates that our result becomes completely consistent with that of Kumhof et al. (2010); namely, equilibrium determinacy is achieved by applying an active monetary policy accompanied by a passive fiscal policy.

[Figure 3]

Furthermore, Benhabib et al. (2003) and Carlstrom and Fuerst (2003) demonstrate that equilibrium indeterminacy can occur even under an active monetary policy in an MIPF model, regardless of whether it postulates discrete time or continuous time. This suggests that the case of Fig. 2 can be considered corresponding to the models proposed by Benhabib et al. (2003) and Carlstrom and Fuerst (2003).[12]

Thus, in the case of no policy lag, the results derived from a discrete time model correspond perfectly to those derived from a continuous time model. In the next section, we examine the case where monetary and fiscal policy lags are present. Specifically, we show that monetary and fiscal authorities can avoid the equilibrium indeterminacy occurring in the case equivalent to an MIPF model ($P_2 > 0$) by utilizing such "time lags."[13] Therefore, in the following discussion, we restrict the analysis to the case where $-v_3 = det J_1 < 0$.

4 Case with positive policy lags

The model economic system when Equations (17) and (20) are used as the interest rate rule and tax rate rule, respectively, can be expressed as follows:

$$
\begin{aligned}
\dot{\mu}_1(t) &= [\rho - R(\pi(t-\theta_1)) + \pi(t)]\mu_1(t),\\
\dot{\pi}(t) &= \rho(\pi(t)-\pi^*) - \frac{\phi}{\eta} z^{-(1+\psi)}[c(\mu_1(t), R(\pi(t-\theta_1))) + g]^{1+\psi}\\
&\quad + (1-\tau(a(t-\theta_2)))\frac{\phi-1}{\eta}\mu_1(t)[c(\mu_1(t), R(\pi(t-\theta_1))) + g],\\
\dot{a}(t) &= [R(\pi(t-\theta_1)) - \pi(t)]a(t) - R(\pi(t-\theta_1))m(c(\mu_1(t), R(\pi(t-\theta_1))), \mu_1(t), R(\pi(t-\theta_1)))\\
&\quad - \tau(a(t-\theta_2))[c(\mu_1(t), R(\pi(t-\theta_1))) + g] + g.
\end{aligned}
\tag{24}
$$

This is a differential equation system with two delays.

The steady-state values of System (24) are given in Equation (22). By linearizing System (24) around the steady state, the equations become

$$
\begin{aligned}
\dot{\hat{\mu}}_1(t) &= -[R'\hat{\pi}(t-\theta_1) - \hat{\pi}(t)]\mu_1^*,\\
\dot{\hat{\pi}}(t) &= \rho\hat{\pi}(t) - P_2R'\hat{\pi}(t-\theta_1) + P_1\hat{\mu}_1(t) - \tau'\frac{\phi-1}{\eta}\mu_1^*[c(\mu_1^*, \bar{R}) + g]\hat{a}(t-\theta_2),\\
\dot{\hat{a}}(t) &= P_3\hat{\mu}_1(t) - a^*\hat{\pi}(t) + (P_4 + a^*)\hat{\pi}(t-\theta_1) + r^*\hat{a}(t) - \tau'[c(\mu_1^*, \bar{R}) + g]\hat{a}(t-\theta_2),
\end{aligned}
\tag{25}
$$

where $\hat{\mu}_1(t) \equiv \mu_1(t) - \mu_1^*$, $\hat{\pi}(t) \equiv \pi(t) - \pi^*$, and $\hat{a}(t) \equiv a(t) - a^*$. Assuming the exponential functions $\hat{\mu}_1(t) = C_1e^{\lambda t}$, $\hat{\pi}(t) = C_2e^{\lambda t}$, and $\hat{a}(t) = C_3e^{\lambda t}$ (where C_1, C_2, and C_3 are arbitrary constants and λ is an eigenvalue) as the solutions to this system and plugging these functions into System (25), we obtain the following:

$$
\begin{bmatrix}
\lambda & -\mu_1^* + R'\mu_1^*e^{-\theta_1\lambda} & 0\\
-P_1 & \lambda - \rho + P_2R'e^{-\theta_1\lambda} & \tau'\frac{\phi-1}{\eta}\mu_1^*y^*e^{-\theta_2\lambda}\\
-P_3 & a^* - (P_4 + a^*)R'e^{-\theta_1\lambda} & \lambda - r^* + \tau'y^*e^{-\theta_2\lambda}
\end{bmatrix}
\begin{bmatrix}
\hat{\mu}_1(t)\\ \hat{\pi}(t)\\ \hat{a}(t)
\end{bmatrix}
=
\begin{bmatrix}
0\\0\\0
\end{bmatrix}.
$$

The determinant of the matrix on the left-hand side, which we denote as $\Delta_2(\lambda)$, should be zero for non-trivial solutions to exist: that is,

$$
\Delta_2(\lambda) \equiv s_0(\lambda) + s_1(\lambda)e^{-\theta_1\lambda} + s_2(\lambda)e^{-\theta_2\lambda} + s_3(\lambda)e^{-(\theta_1+\theta_2)\lambda} = 0, \tag{26}
$$

where

$$
\begin{aligned}
s_0(\lambda) &= \lambda^3 - (\rho + r^*)\lambda^2 + (\rho r^* - P_1\mu_1^*)\lambda + P_1\mu_1^* r^*,\\
s_1(\lambda) &= P_2R'\lambda^2 - P_2R'r^*\lambda + P_1R'\mu_1^*\lambda - P_1R'\mu_1^* r^*,\\
s_2(\lambda) &= \tau' y^*\lambda^2 - \rho\tau' y^*\lambda - a^*\tau'\frac{\phi-1}{\eta}\mu_1^* y^*\lambda + P_3\mu_1^{*2}\tau'\frac{\phi-1}{\eta}y^* - P_1\mu_1^*\tau' y^*,\\
s_3(\lambda) &= P_2R'\tau' y^*\lambda + (P_4 + a^*)R'\tau'\frac{\phi-1}{\eta}\mu_1^* y^*\lambda - P_3R'\mu_1^{*2}\tau'\frac{\phi-1}{\eta}y^* + P_1R'\mu_1^*\tau' y^*.
\end{aligned}
$$

Equation (26) is the characteristic equation for System (25).

Equation (26) has an infinite number of roots owing to the existence of terms including the exponential functions $e^{-\theta_1\lambda}$, $e^{-\theta_2\lambda}$, and $e^{-(\theta_1+\theta_2)\lambda}$.[14] In addition, unlike ordinary differential equations, delay-differential equations require initial values that are evaluated not only in time $t = t_0$ (present time) but also in time $t_0 - \theta_1 \le t < t_0$ and $t_0 - \theta_2 \le t < t_0$ (past times). As $\mu_1(t)$ and $\pi(t)$ are the jump variables, their initial values should be determined by economic agents. However, in time t_0, they can only determine the values $\mu_1(t_0)$ and $\pi(t_0)$, because past values for these variables should be considered as given. Therefore, if there are exactly two roots with positive real parts among the infinite number of roots, then the initial values are uniquely determined; that is, the equilibrium is locally determinate. However, the equilibrium will be indeterminate if less than two roots have positive real parts, and the equilibrium will become unstable if more than two roots have positive real parts (an equilibrium will not be present).

A numerical method developed by Lin and Wang (2012) can be used to investigate an equation that includes exponential functions, as in Equation (26).[15]

4.1 Preconditions

To apply Lin and Wang's (2012) method, some preconditions need to be examined; that is, Equation (26) should satisfy the following conditions:

(i) $\deg(s_0(\lambda)) \ge \max\{\deg(s_1(\lambda)), \deg(s_2(\lambda)), \deg(s_3(\lambda))\}$;

(ii) $\Delta_2(0) \ne 0$;

(iii) A solution common to all four polynomials $s_0(\lambda) = 0$, $s_1(\lambda) = 0$, $s_2(\lambda) = 0$, and $s_3(\lambda) = 0$ does not exist (i.e., these are coprime);

(iv) $\lim_{\lambda\to\infty}(|s_1(\lambda)/s_0(\lambda)| + |s_2(\lambda)/s_0(\lambda)| + |s_3(\lambda)/s_0(\lambda)|) < 1$.

While Condition (i) is satisfied by $3 > \max\{2, 2, 1\}$, Condition (ii) is satisfied by $\Delta_2(0) = P_1\mu_1^*(r^* - \tau' y^*)(1 - R') + P_3\mu_1^{*2}\tau'\frac{\phi-1}{\eta}y^*(1 - R') = v_3 = -detJ_1 \neq 0$. Condition (iii) can numerically be confirmed as follows: values that satisfy $s_0(\lambda) = 0$ are calculated as $\lambda = -0.103,\ 0.005,\ 0.108$. These values cannot be the solutions to $s_3(\lambda) = 0$ for $\{(R', \tau') : R' \in (0, 3), \tau' \in (0, 0.1)\}$. Therefore, Condition (iii) is satisfied. Finally, Condition (iv) is satisfied by $\lim_{\lambda\to\infty}(|s_1(\lambda)/s_0(\lambda)| + |s_2(\lambda)/s_0(\lambda)| + |s_3(\lambda)/s_0(\lambda)|) = 0$.

Now, we examine the effects of lags (θ_1, θ_2) on local equilibrium determinacy. The procedure for the analysis is as follows:

(1) The points where pure imaginary roots appear, that is, the points where the dynamics can change, are characterized (if they are present).[16] These points are referred to as the "crossing points."

(2) We describe the sets of crossing points as the "crossing curves" on the $(\theta_1, \theta_2) \in \mathbb{R}^2_+$ plane using numerical simulation.

(3) We indicate the existence of regions where local determinacy is achieved.

4.2 Crossing curves

We denote a pure imaginary root as $\lambda = i\omega$ (where ω = imaginary part[17] > 0, and $i = \sqrt{-1}$). By plugging this expression into Equation (26), we obtain the following:

$$\Delta_2(i\omega) = s_0(i\omega) + s_1(i\omega)e^{-i\omega\theta_1} + s_2(i\omega)e^{-i\omega\theta_2} + s_3(i\omega)e^{-i\omega(\theta_1+\theta_2)} = 0. \quad (27)$$

We first characterize the values of ω that satisfy Equation (27). According to Lemma 3.2 in Lin and Wang (2012), $\Delta_2(i\omega) = 0$ holds for $\omega \in \mathbb{R}_+$ satisfying the following:

$$F(\omega) \equiv (|s_0|^2 + |s_1|^2 - |s_2|^2 - |s_3|^2)^2 - 4(M_1^2 + N_1^2) < 0,$$

where

$$M_1(\omega) = \mathrm{Re}(s_2\bar{s}_3) - \mathrm{Re}(s_0\bar{s}_1),$$
$$N_1(\omega) = \mathrm{Im}(s_2\bar{s}_3) - \mathrm{Im}(s_0\bar{s}_1).$$

We denote the set of $\omega > 0$ that satisfies $F(\omega) < 0$ as Ω (crossing frequency set). For $\omega \in \Omega$, the sets (θ_1, θ_2) satisfying Equation (27) (crossing points) can be expressed as follows (see Equation 17 in Lin and Wang, 2012):

$$\begin{aligned}\Theta^{\pm} &\equiv \{(\theta_1^{\pm}(\omega), \theta_2^{\mp}(\omega)) \in \mathbb{R}_+^2\} \\ &= \left\{ \left(\frac{\pm\delta_1(\omega) - \varphi_1(\omega) + 2n_1\pi}{\omega}, \frac{\mp\delta_2(\omega) - \varphi_2(\omega) + 2n_2\pi}{\omega} \right);\ n_1, n_2 \in \mathbb{Z} \right\},\end{aligned} \tag{28}$$

where

$$\begin{aligned}\delta_1(\omega) &= \cos^{-1}\left(\frac{|s_0|^2 + |s_1|^2 - |s_2|^2 - |s_3|^2}{2\sqrt{M_1^2 + N_1^2}} \right);\ \delta_1 \in [0, \pi], \\ \varphi_1(\omega) &= \arg\{s_2\bar{s}_3 - s_0\bar{s}_1\} \\ &= \tan^{-1}\left(\frac{\mathrm{Im}(s_2\bar{s}_3 - s_0\bar{s}_1)}{\mathrm{Re}(s_2\bar{s}_3 - s_0\bar{s}_1)} \right), \\ \delta_2(\omega) &= \cos^{-1}\left(\frac{|s_0|^2 - |s_1|^2 + |s_2|^2 - |s_3|^2}{2\sqrt{M_2^2 + N_2^2}} \right);\ \delta_2 \in [0, \pi], \\ M_2(\omega) &= \mathrm{Re}(s_1\bar{s}_3) - \mathrm{Re}(s_0\bar{s}_2), \\ N_2(\omega) &= \mathrm{Im}(s_1\bar{s}_3) - \mathrm{Im}(s_0\bar{s}_2), \\ \varphi_2(\omega) &= \arg\{s_1\bar{s}_3 - s_0\bar{s}_2\} \\ &= \tan^{-1}\left(\frac{\mathrm{Im}(s_1\bar{s}_3 - s_0\bar{s}_2)}{\mathrm{Re}(s_1\bar{s}_3 - s_0\bar{s}_2)} \right).\end{aligned}$$

Lin and Wang (2012) also demonstrate that Θ^+ and Θ^- form a class of continuous curves on $\mathbb{R}_+^2$. We call these curves crossing curves. In the next subsection, we illustrate an example of crossing curves using a numerical simulation.

4.3 Numerical simulation

We assume the same parameter values and functional form of the utility function as in the previous section. Furthermore, we suppose that the monetary authorities implement an active policy ($R' = 1.5$) and the fiscal authority implements a passive policy ($\tau' = 0.09$). Then, if monetary and fiscal policy lags are not present (i.e., $\theta_1 = \theta_2 = 0$), indeterminacy occurs, wherein the signs of the roots are ($---$), as shown in the previous section.

The crossing set Ω is calculated as $\omega \in (0.126, 0.460)$ (Fig. 4). For $\omega \in \Omega$, we can describe Θ^+ and Θ^- as shown in Fig. 5. The solid curves represent Θ^+, and the dashed curves represent Θ^-.

[Figure 4]

[Figure 5]

We call the direction of the curve corresponding to increasing ω as "positive direction." When we move in the positive direction along curves Θ^+ (Θ^-), the region on the left-hand side of Θ^+ (Θ^-) has two more (less) roots with positive real parts (Theorem 4.1 in Lin and Wang, 2012). In Fig. 5, we use arrows to indicate the crossing directions to which roots with positive real parts increase when lags (θ_1, θ_2) intersect with these curves. The region on the end of an arrow has two more roots with positive real parts.

In the three regions indicated by D_1, D_2, and D_3 in Fig. 5, exactly two roots with positive real parts exist: therefore, the equilibrium is locally determinate. Thus, we have the following proposition:

Proposition 4.1 Monetary and fiscal authorities can establish local equilibrium determinacy by introducing lags into policy responses when the equilibrium is indeterminate and where the signs of the roots are ($---$) under policies without lags.

This proposition indicates that desirable combinations of monetary and fiscal policy lags exist. If lags are too short, then equilibrium indeterminacy will occur, and if they are too long, the equilibrium will become unstable.

Moreover, under any configuration of crossing curves, the number of roots with positive real parts changes by two at a time when θ_1 and θ_2 intersect a crossing curve. Hence, it is impossible for monetary and fiscal authorities to achieve local determinacy by policy lags if the signs of the roots are ($+--$) under policies without a lag.

5 Conclusion

In this paper, we examined the effects of monetary and fiscal policies on local equilibrium determinacy by developing a continuous-time NK model introducing delays into policy responses. We demonstrated that the responsiveness of policy variables to economic fluctuations as well as timings of their implementation has a significant role in achieving local determinacy.

On the assumption of a plausible parameter set, indeterminacy can arise under the combination of active monetary and passive fiscal policies when no delays occur in policy implementation by the monetary and fiscal authorities. However, such indeterminacy can be resolved by setting lags in the desirable regions, as indicated by D_1–D_3 in Fig. 5. This finding suggests that the monetary and fiscal authorities may have to "purposefully" delay their policies.

We also point that monetary and fiscal authorities have complementarity in their reaction rates. That is, if the monetary policy lag is too short and indeterminacy occurs, determinacy can be established by lengthening the fiscal policy lag, and vice versa.

In any case, policy lags can have either stabilizing or destabilizing effects on an economy.

Notes

1 The introductory textbooks for NK models include Woodford (2003), Walsh (2010), and Galí (2015).

2 One method to treat the "past" of a variable in continuous-time models is to employ a weighted average of such a variable's stream that extends from the infinite past to the present, as in Benhabib et al. (2003). However, a time lag in the strict sense cannot be represented by a variable's weighted average, but should be represented by a past value taken at a certain point in time.

3 Friedman (1948) is a pioneering work on policy lags.

4 See Dixit and Stiglitz (1977).

5 Heterogeneity of goods is reflected by the assumption of $\phi > 1$. If the goods are completely homogeneous (complete substitution), $\phi \to 1$.

6 See Blanchard and Kiyotaki (1987) for details.

7 The price revision cost is specified in a quadratic equation consistent with that outlined by Rotemberg (1982). In addition, this cost can be interpreted as the psychological stress due to price negotiations.

8 In Fig. 1, $\theta_1 = 2.0$ and $t = 0$.

9 Equation (22) is used to derive the expressions of P_1 and P_2.

10 See Chapter 18 in Gandolfo (2010).

11 In the case of distortionary taxes, as the income tax, the dynamic system becomes indecomposable; that is, the law of motion of π is affected by a. Thus, the bifurcation value of τ' (0.019) found in Fig. 2 slightly exceeds the value of $r^*/y^* = 0.013$.

12 The main contribution of Benhabib et al. (2003) was the finding that a Hopf bifurcation can occur under an active monetary policy, indicating the presence of *global* indeterminacy. However, we emphasize here the point that an active monetary policy and local indeterminacy are not mutually exclusive.

13 Time lags can increase the number of roots with positive real parts. In models without a micro-foundation, as those proposed in Asada and Yoshida (2001) and Yoshida and Asada (2007), this change implies destabilization. However, in our model, the change can achieve determinacy, implying that time lags can stabilize an economy.

14 See Chapter 3 in Bellman and Cooke (1963) for details.

15 The merit of Lin and Wang's (2012) method is that it can be applied to the case where an equation includes not only exponential functions as $e^{-\theta_1\lambda}$ and $e^{-\theta_2\lambda}$ but also the function $e^{-(\theta_1+\theta_2)\lambda}$. If $s_3(\lambda) = 0$, then we can use a different method developed by Gu et al. (2005).

16 Condition (ii) ensures that zero cannot be a root.

17 We can assume that $\omega > 0$ without losing generality because imaginary roots are necessarily conjugated.

References

[1] Asada, Toichiro, and Hiroyuki Yoshida (2001). "Stability, instability and complex behavior in macrodynamic models with policy lag," *Discrete Dynamics in Nature and Society* 5(4), 281–295.

[2] Bellman, Richard Ernest, and Kenneth L. Cooke (1963).*Differential-Difference Equations*, Academic Press, New York.

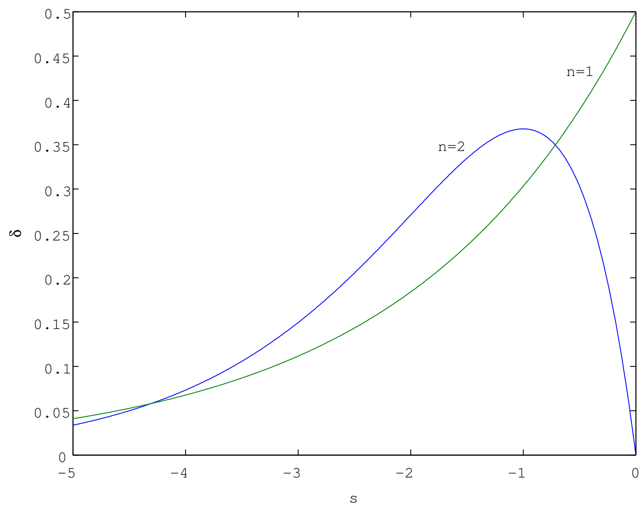

Figure 1: Function $\delta(s)$

[3] Benhabib, Jess, Stephanie Schmitt-Grohé, and Martin Uribe (2003). "Backward-looking interest-rate rules, interest-rate smoothing, and macroeconomic instability," *Journal of Money, Credit and Banking* 35(6), 1379–1412.

[4] Bilbiie, Florin O. (2008). "Limited asset markets participation, monetary policy and (inverted) aggregate demand logic," *Journal of Economic Theory* 140(1), 162–196.

[5] Blanchard, Olivier Jean, and Nobuhiro Kiyotaki (1987). "Monopolistic competition and the effects of aggregate demand," *American Economic Review* 77(4), 647–666.

[6] Carlstrom, C. T. and T. S. Fuerst (2003). "Comment on 'Backward-Looking Interest-Rate Rules, Interest-Rate Smoothing, and Macroeconomic Instability' by Jess Benhabib," *Journal of Money, Credit and Banking* 35, 1413–1423.

[7] Dixit, Avinash K., and Joseph E. Stiglitz (1977). "Monopolistic competition and optimum product diversity," *American Economic Review* 67(3), 297–308.

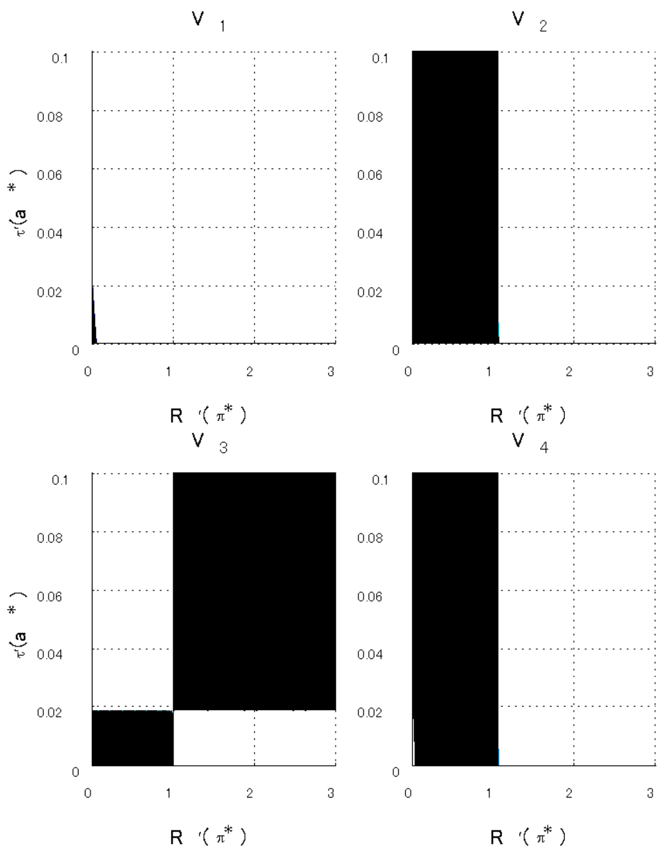

Figure 2: Case in which $\sigma = 2$

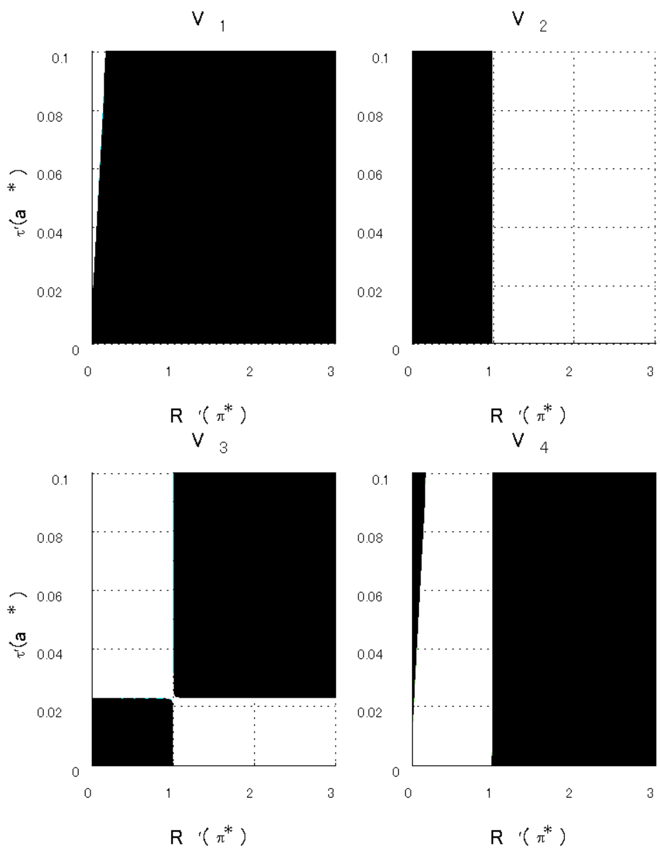

Figure 3: Case in which $\sigma = 0.9$

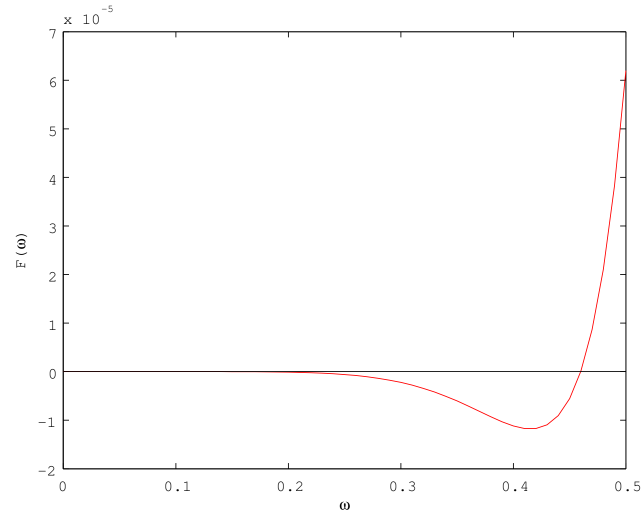

Figure 4: Graph of $F(\omega)$

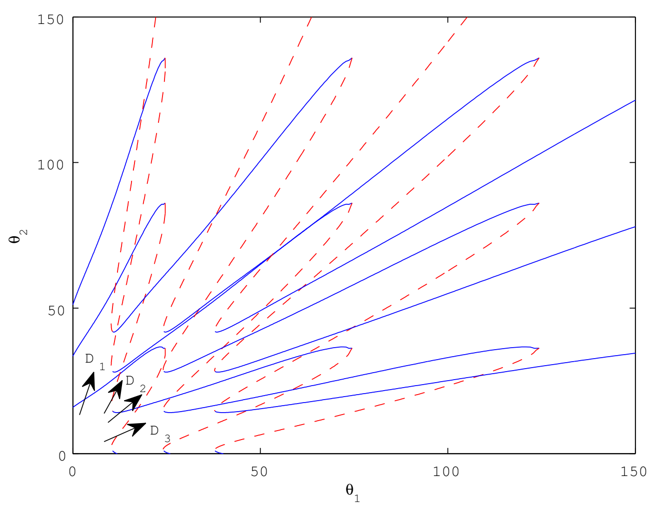

Figure 5: Crossing curves

[8] Feenstra, Robert C. (1986). "Functional equivalence between liquidity costs and the utility of money," *Journal of Monetary Economics* 17(2), 271–291.

[9] Friedman, Milton (1948). "A monetary and fiscal framework for economic stability," *American Economic Review* 38(3), 245–264.

[10] Galí, Jordi (2015). *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications*, Princeton, NJ: Princeton University Press.

[11] Gandolfo, Giancarlo (2010). *Economic Dynamics*, Fourth Edition, Berlin: Springer-Verlag.

[12] Gliksberg, Baruch (2009). "Monetary policy and multiple equilibria with constrained investment and externalities," *Economic Theory* 41(3), 443–463.

[13] Gu, Keqin, Silviu-Iulian Niculescu, and Jie Chen (2005). "On stability crossing curves for general systems with two delays," *Journal of Mathematical Analysis and Applications* 311(1), 231–253.

[14] Kumhof, Michael, Ricardo Nunes, and Irina Yakadina (2010). "Simple monetary rules under fiscal dominance," *Journal of Money, Credit and Banking* 42(1), 63–92.

[15] Leeper, Eric M. (1991). "Equilibria under 'active' and 'passive' monetary and fiscal policies," *Journal of Monetary Economics* 27(1), 129–147.

[16] Lin, Xihui, and Hao Wang (2012). "Stability analysis of delay differential equations with two discrete delays," *Canadian Applied Mathematics Quarterly* 20(4), 519–533.

[17] Meng, Qinglai, and Chong K. Yip (2004). "Investment, interest rate rules, and equilibrium determinacy," *Economic Theory* 23(4), 863–878.

[18] Rotemberg, Julio J. (1982). "Sticky prices in the United States," *Journal of Political Economy* 90(6), 1187–1211.

[19] Schmitt-Grohé, Stephanie, and Martin Uribe (2007). "Optimal simple and implementable monetary and fiscal rules," *Journal of Monetary Economics* 54(6), 1702–1725.

[20] Tsuzuki, Eiji (2014). "A New Keynesian model with delay: Monetary policy lag and determinacy of equilibrium," *Economic Analysis and Policy* 44(3), 279–291.

[21] Tsuzuki, Eiji (2015). "Determinacy of equilibrium in a New Keynesian model with monetary policy lag," *International Journal of Economic Behavior and Organization*, Special Issue: Recent Developments of Economic Theory and Its Applications, 3 (2-1), 15–22.

[22] Walsh, Carl E. (2010).*Monetary Theory and Policy*, Third Edition, Cambridge, MA: MIT Press.

[23] Woodford, Michael (2003).*Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press.

[24] Yoshida, Hiroyuki, and Toichiro Asada (2007). "Dynamic analysis of policy lag in a Keynes–Goodwin model: stability, instability, cycles and chaos," *Journal of Economic Behavior and Organization* 62(3), 441–469.