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WASC STOCHASTIC VOLATILITY JUMP

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Abstract. In this article, we develop a model which estimate the value of a basket of the underlying assets. The aim of this paper is to obtain a new form of a model, the stylized facts of WASC model and consider the risk created by the turbulence effect on the market which causes by the abrupt and sudden movements in underlying prices. In this case, we transform WASC and change the O.U-type process used in the WASC model into a Levy process in $\mathbb{R}^n$ with jumps below. Our task is then to find the conditions on the parameters of the model by regularizing the volatility in a way so that it remains positive definite even if it jumps, capturing the stylized facts of WASC model and obtaining the closed form expression of the characteristic functions of the model.

1. Introduction

Famous on the stylized facts, WASC is a reference for a multidimensional model to evaluate the value of a basket of the underlying assets. Moreover, Multivariate Stochastic Volatility Models of O.U type gives account of the risk created by the turbulence effect on the market which is not captured by WASC. Indeed, The recent perturbations in the financial market induce unexpected and unpredictable events on the assets prices (rare occurrences) which are difficult to capture by a continuous model or WASC because the value of volatility of Γ_t expected is mostly big (there is an anomaly). So, we want elaborate a new multidimensional model which gives account all the risk created by the perturbation on the market and the stylized facts of the WASC model at the same time.
To explain this risk created by the turbulence effect on the market, we introduce a jumps process ψ_t in the model dynamic of volatility. To do this, we change the O.U-type process in $\mathbb{R}^n$ used in the WASC model $dx_t = \Phi x_t dt + Q dW_t$ into a Levy process in $\mathbb{R}^n$

$$dx_t = \Phi x_t dt + \sqrt{Q'Q} dW_t + dP_t \tag{1.1}$$

where

- Φ and Q are $n \times n$ dimensional real matrices

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- W_t is a n-dimensional stochastic vector whose components are standard Brownian motions and
- P_t is a n-dimensional vector of compound Poisson process.

Thus, we obtain the differential $d\psi_t$ above by deriving the volatility of the form :

$$z_t = \sum_{i=1}^{\nu} x_{i,t}(x_{i,t})' \tag{1.2}$$

where ν is a positive integer nonzero and $(x_{i,t})_t$ $i = 1, ..., \nu$ are the n dimensional vector of process defined by (1.1).

How to transmit the jumps of the volatility towards the yield? The Multivariate Stochastic Volatility Models of O.U type inspires us to introduce the parameter φ by producing with the jumps process of the volatility ψ_t and we obtain a model of the form:

$$\begin{cases} d\log S_t = \left(\mu + \begin{bmatrix} tr(D_1\Gamma_t) \\ : \\ tr(D_n\Gamma_t) \end{bmatrix}\right) dt + \sqrt{\Gamma_t} dZ_t + d\psi_t\varphi \\ d\Gamma_t = (\nu Q'Q + \Phi\Gamma_t + \Gamma_t\Phi')dt + \sqrt{\Gamma_t} d\tilde{W}_t\sqrt{Q'Q} + \sqrt{Q'Q}(d\tilde{W}_t)'\sqrt{\Gamma_t} + d\psi_t \\ dZ_t = \sqrt{1-\rho'\rho}dB_t + d\tilde{W}_t\rho \\ d\psi_t = \sqrt{\Gamma_t}d\tilde{P}_t + (d\tilde{P}_t)'\sqrt{\Gamma_t} + (d\tilde{P}_t)(d\tilde{P}_t)' \end{cases} \tag{1.3}$$

with

- ν is a positive integer nonzero;
- φ and μ are vectors in $\mathbb{R}^n$;
- Q and Φ are $n \times n$ dimensional real matrices;
- D_i, $i = 1, ...n$ are $n \times n$ dimensional real matrices;
- $dZ_t = \sqrt{1-\rho'\rho}dB_t + d\tilde{W}_t\rho$ defines the stochastic correlation noise between the yield $\log S_t$ and its volatility Γ_t on the continuous part of the trajectory;
- $\rho = (\rho_1, \rho_2, ..., \rho_n)'$ where $\rho_i \in [-1\ ,\ 1]$;
- B_t is a n-dimensional vector whose components are Brownian motions;
- $\tilde{W}_t$ is a $n\times n$ dimensional stochastic matrix whose components are Brownian motions;
- $\tilde{P}_t$ is a $n \times n$ dimensional stochastic matrix whose components are the compounded Poisson processes;
- y' is the transpose of the vector y;
- H' is the transpose of the matrix H; $tr(H)$ is the trace of the matrix H.

Our study is therefore to provide conditions on the parameters of the model so that it can be used to estimate the price of a basket carrying several underlying assets by accounting for the risk created by the turbulence effect on the market and the risks treated by the WASC model.
In the later section, we try out to estimate the values of the indexes CAC40 and SP500 using our model and we estimate the parameters of model by using the C.GMM (Generalized Method of Moments based on the continuum of moment conditions) method based on the historical data.

2. Model

In this section, we will see successively : details of the dynamics which lead to this multidimensional model on the one hand; the characteristics specifying this on

the other hand; and finally, we give the characteristic functions of the model to estimate its parameters.
Let ($\mathbb{R}^n$, $\mathbb{P}$) be a probability space where $\mathbb{P}$ is the "risk-neutral" probability such that the price of any option is a conditional expectation of its future income.
Let us consider a market of a basket carrying n underlying assets.
Let S_t be the value of this basket at time t and $\log S_t$ is its return.

2.1. **Dynamic of volatility.** Let $(x_t)_{t\geq 0}$ be a process in $\mathbb{R}^n$ defined by (1.1) and $(z_t)_{t\geq 0}$ be a form of (1.2).

Proposition 2.1. *z_t is a positive definite matrix if and only if $\nu \geq n \geq 1$.*

Proof. "$\Rightarrow$" This is obvious for $\nu = 1$.
Let us now consider for $\nu \geq 2$.
Using absurd reasoning, suppose that $n > \nu$ and z_t is a positive definite matrix.
Let a $n \times \nu$ dimensional process be :

$$d(X_t)' = \Phi(X_t)'dt + \sqrt{Q'Q}d\check{W}_t + (d\check{P}_t)', \tag{2.1}$$

where

- $(X_t)' = (x_{1,t}, ..., x_{\nu,t})$ is a $n \times \nu$ dimensional stochastic matrix;
- $\check{W}_t = (W_{1t}, W_{2,t}, ..., W_{\nu,t})$ is the $n \times \nu$ dimensional matrix where the $W_{i,t}$ are the Brownian motion vectors of $x_{i,t}$, $i = 1, ..., \nu$;
- $(\check{P}_t)' = (P_{1t}, P_{2,t}, ..., P_{\nu,t})$ is the $n \times \nu$ dimensional matrix where the $P_{i,t}$ are n-dimensional vectors of compound Poisson process of $x_{i,t}$, $i = 1, ..., \nu$.

We have $z_t = (X_t)'X_t$. So $rank(z_t) \leq min(n, \nu)$. And since $\nu < n$, then we have $rank(z_t) < n$.
In addition, as z_t is a $n \times n$ dimensional matrix, then z_t is singular and therefore it is not positive definite. A contradiction with z_t is positive definite matrix.
"$\Leftarrow$" Suppose that $\nu \geq n \geq 1$ and we show that z_t is positive definite matrix.
Let $y \in \mathbb{R}^p$, $y = (y_1, y_2, ..., y_p)'$. If we develop z_t of the form (1.2) and assume $x_{i,t} = (x^i_1, ..., x^i_n)'$, we have

$$z_t = \sum_{i=1}^{\nu} \begin{bmatrix} (x^i_1)^2 & x^i_1 x^i_2 & \cdots & x^i_1 x^i_n \\ x^i_2 x^i_1 & (x^i_2)^2 & \cdots & x^i_2 x^i_n \\ \vdots & \vdots & \cdots & \vdots \\ x^i_n x^i_1 & x^i_n x^i_2 & \cdots & (x^i_n)^2 \end{bmatrix}.$$

So

$$\begin{aligned} y' z_t y &= \sum_{j=1}^{n} y_j y_1 \sum_{i=1}^{\nu} x^i_1 x^i_j + \sum_{j=1}^{n} y_j y_2 \sum_{i=1}^{\nu} x^i_2 x^i_j + \cdots + \sum_{j=1}^{n} y_j y_n \sum_{i=1}^{\nu} x^i_n x^i_j \\ &= \sum_{i=1}^{\nu}\sum_{j=1}^{n} y_j y_1 x^i_1 x^i_j + \sum_{i=1}^{\nu}\sum_{j=1}^{n} y_j y_2 x^i_2 x^i_j + \cdots + \sum_{i=1}^{\nu}\sum_{j=1}^{n} y_j y_n x^i_n x^i_j \\ &= \sum_{i=1}^{\nu}\sum_{j=1}^{n} (y_j x^i_j)^2 + \sum_{i=1}^{\nu} \sum_{\substack{k,l=1 \\ k\neq l}}^{n} y_k y_l x^i_k x^i_l \\ &= \sum_{i=1}^{\nu} \left(\sum_{j=1}^{n} (y_j x^i_j)^2 + \sum_{\substack{k,l=1 \\ k\neq l}}^{n} y_k y_l x^i_k x^i_l \right) \end{aligned}$$

$$\begin{aligned} &= \sum_{i=1}^{\nu}\left(\sum_{j=1}^{n}(y_j x_j^i)\right)^2 \\ &\geq 0. \end{aligned}$$

Thus, if the latter is zero, we obtain ν equations with n unknowns, which are :

$$\begin{cases} \sum\limits_{j=1}^{n}(y_j x_j^1) = 0 \\ \sum\limits_{j=1}^{n}(y_j x_j^2) = 0 \\ \vdots \\ \sum\limits_{j=1}^{n}(y_j x_j^\nu) = 0. \end{cases}$$

Since $\nu \geq n$, then the unknown $y_j = 0$ for all j.
Thus, the later is strictly positive for all $0 \neq y \in \mathbb{R}^n$ and it follows that Γ_t is positive definite matrix. □

Let $X = (X_t)_{t\geq 0}$ be a Levy process. We define the jump process associated with X by

$$\Delta X = (\Delta X_t \ ; t \geq 0) \tag{2.2}$$

with $\Delta X_t = X_t - X_{t^-}$ where $X_{t^-} = \lim\limits_{s\to t^-} X_s$.
Let $X = (X_t)_{t\geq 0}$ be a Levy process defined by :

$$X_t := X_0 + \int_0^t K(s)ds + \int_0^t \varphi(s)dW_s + \Delta X_t \tag{2.3}$$

with K and φ are real processes such that for all $t \geq 0$; $\int_0^t \mid K(s) \mid ds < +\infty$ and $\int_0^t \mid \varphi(s) \mid^2 ds < +\infty$ p.s; ΔX_t defines the jumps process of X_t and W_t is a Brownian motion. We define X_t^c the part of X_t defined by :

$$X_t^c := X_0 + \int_0^t K(s)ds + \int_0^t \varphi(s)dW_s \text{ for all } t \geq 0. \tag{2.4}$$

The part X_t^c of X_t is called the continuous part of X_t.

Proposition 2.2. *If $\nu \geq n$, then the process z_t satisfies the SDE (Stochastic Differential Equation) of type :*

$$\begin{aligned} dz_t = (\nu Q'Q + \Phi z_t + z_t\Phi')dt + \sqrt{z_t}(d\tilde{W}_t)'\sqrt{Q'Q} + \sqrt{Q'Q}d\tilde{W}_t\sqrt{z_t} + \sqrt{z_t}(d\tilde{P}_t)' \\ + d\tilde{P}_t\sqrt{z_t} + d\tilde{P}_t(d\tilde{P}_t)' \end{aligned} \tag{2.5}$$

with $\tilde{W}_t$ is a $n\times n$ dimensional stochastic matrix whose components are independent Brownian motions; Q and Φ are the above matrices ; $(\tilde{P}_t)$ is a $n \times n$ dimensional stochastic matrix whose components are the compounded Poisson processes.

Proof. Let us $dP_{i,t} = Y_i dN_t$ where the Y_i are n-dimensional vectors of i.i.d (independent and identically distributed) random variables and N_t is a Poisson process of intensity $\lambda > 0$. Applying Ito's formula on the Levy process $f(x_t) = \sum\limits_{i=1}^{\nu} x_{i,t}(x_{i,t})'$, we obtain

$$dz_t = \sum_{i=1}^{\nu} dx_{i,t}^c(x_{i,t}^c)' + \sum_{i=1}^{\nu} x_{i,t}(dx_{i,t}^c)' + \frac{1}{2}\times 2\sum_{i=1}^{\nu} < dx_{i,t}^c, (dx_{i,t}^c)' >$$

$$
\begin{aligned}
& + \left[\sum_{i=1}^{\nu} (x_{i,t} + Y_i)(x_{i,t} + Y_i)' - x_{i,t}(x_{i,t})' \right] dN_t \\
= \; & \sum_{i=1}^{\nu} (\Phi x_{i,t} dt + \sqrt{Q'Q} dW_{i,t})(x_{i,t})' + \sum_{i=1}^{\nu} x_{i,t}(\Phi x_{i,t} dt + \sqrt{Q'Q} dW_{i,t})' \\
& + \sum_{i=1}^{\nu} \sqrt{Q'Q} dW_{i,t}(dW_{i,t})' \sqrt{Q'Q} + [x_{i,t}(x_{i,t})' + x_{i,t}(Y_i)' + Y_i(x_{i,t})' \\
& + Y_i(Y_i)' - x_{i,t}(x_{i,t})'] dN_t \\
= \; & (\nu Q'Q + \Phi z_t + z_t \Phi')dt + \sum_{i=1}^{\nu} \sqrt{Q'Q} dW_{i,t}(x_{i,t})' + x_{i,t}(dW_{i,t})' \sqrt{Q'Q} \\
& + \sum_{i=1}^{\nu} [x_{i,t}(Y_i)' + Y_i(x_{i,t})' + Y_i(Y_i)'] dN_t \\
= \; & (\nu Q'Q + \Phi z_t + z_t \Phi')dt + \sqrt{Q'Q} d\check{W}_t X_t + (X_t)'(d\check{W}_t)' \sqrt{Q'Q} + (X_t)' d\check{P}_t \\
& + (d\check{P}_t)' X_t + (d\check{P}_t)' d\check{P}_t. \qquad (2.6)
\end{aligned}
$$

Since z_t is a positive definite matrix through the Proposition 2.1, then let us assume $d\tilde{P}_t = (d\check{P}_t)' X_t(\sqrt{z_t})^{-1}$ and $d\tilde{W}_t = d\check{W}_t X_t(\sqrt{z_t})^{-1}$. We have $d\tilde{P}_t(d\tilde{P}_t)' = (d\check{P}_t)' X_t (z_t)^{-1} (X_t)' d\check{P}_t = (d\check{P}_t)' d\check{P}_t$, because $X_t(z_t)^{-1}(X_t)' = I_\nu$ where I_ν is the $\nu \times \nu$ dimensional identity matrix. Indeed, let us look for a $\nu \times \nu$ dimensional real matrix y such that $y = X_t(z_t)^{-1}(X_t)'$.
We have $(X_t)' y = I_n (X_t)' = (X_t)'$. Hence $y = I_\nu$.
So, (2.6) $= (\nu Q'Q + \Phi z_t + z_t \Phi')dt + \sqrt{Q'Q} d\tilde{W}_t \sqrt{z_t} + \sqrt{z_t}(d\tilde{W}_t)' \sqrt{Q'Q} + \sqrt{z_t}(d\tilde{P}_t)' + d\tilde{P}_t \sqrt{z_t} + d\tilde{P}_t (d\tilde{P}_t)'$. □

In the following sections, the stochastic volatility of the model Γ_t is a solution of the SDE defined by (2.5).

2.2. **Dynamic of asset return and Correlation.** The dynamic of asset return represented in (1.3) is based on the model Gourrieroux and Suffana (see the reference [20]) by introducing the jumps process of volatility ψ_t. Producing ψ_t with the vector $\varphi \in \mathbb{R}^n$, the yield jumps according to its volatility. In this case, φ is the frequency and the direction of the yield jumps.
On the continuous part, the stochastic process Z_t defined in (1.3) is allow us to get the asymmetric correlation between the yield and its volatility.
We will see later the conditions on the parameters to obtain these asymmetric correlations.

3. Characteristic functions of the model

In this section, we try to give the explicit expressions of characteristic functions yield and its volatility.
Let us recall the Feynmann-Kac argument following (see the reference [10]):
let $X = (X_t)_{t \geq 0}$ be a Levy process solution of the SDE :

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t + G(X_t)dN_t \qquad (3.1)$$

where the functions b, σ and G are measurable and N_t is a Poisson process of intensity $\lambda > 0$. The solution of $F(h, X_t) = \mathbb{E}\{e^{\int_t^{t+h} g(X_v)dv} f(X_{t+h})/X_t\}$ for all $t, h \geq 0$ where $f, g \in \mathcal{C}^2$ is determined by $\mathcal{L}F(h, X_t) = g(X_t)F(h, X_t)$ with $\mathcal{L}$ is the operator of infinitesimal generator defined by:

$$\mathcal{L}F(h, X_t) =< b(X_t), \frac{\partial F(h, X_t)}{\partial X_t} > + \frac{1}{2} tr\left[\sigma(X_t)\sigma'(X_t)\frac{\partial^2 F(h, X_t)}{\partial X_t^2}\right] - \frac{\partial F(h, X_t)}{\partial h} + \lambda \mathbb{E}\left\{F(h, X_{t^-} + G(X_{t^-})) - F(h, X_{t^-})/X_t\right\} \quad (3.2)$$

with the boundary condition $F(h, X_{t+h}) = f(X_{t+h})$.

3.1. **Characteristic function of asset returns.** When there exist a no-arbitrage opportunity in the market, the model checks:

$$\begin{cases} d\log S_t = (r\check{1} - \frac{1}{2}vec[tr(e_{ii}\Gamma_t)])dt + \sqrt{\Gamma_t}dZ_t + d\psi_t\varphi \\ d\Gamma_t = (\nu Q'Q + \Phi\Gamma_t + \Gamma_t\Phi')dt + \sqrt{\Gamma_t}d\tilde{W}_t\sqrt{Q'Q} + \sqrt{Q'Q}(d\tilde{W}_t)'\sqrt{\Gamma_t} + d\psi_t \\ dZ_t = \sqrt{1-\rho'\rho}dB_t + d\tilde{W}_t\rho \\ d\psi_t = \sqrt{\Gamma_t}d\tilde{P}_t + (d\tilde{P}_t)'\sqrt{\Gamma_t} + (d\tilde{P}_t)(d\tilde{P}_t)' \end{cases} \quad (3.3)$$

where

- $\check{1}$ is a n-dimensional vector whose components are equal to 1;
- If $a_1, ..., a_n \in \mathbb{R}$, we define $vec(a_i) = (a_1, ..., a_n)'$ which is a vector in $\mathbb{R}^n$;
- e_{ii} is the $n \times n$ dimensional matrix defined by $e_{ii} = (\delta_{ijk})_{j,k=1...n}$ where $\delta_{ijk} = \begin{cases} 1 \text{ if } (j,k) = (i,i) \\ 0 \text{ otherwise} \end{cases}$.

Let us $d\tilde{P}_t = JdN_t$ with $J = (J_{lk})_{1\leq l,k\leq n}$ where J_{lk} are the i.i.d normal random variables with $J_{lk} \rightsquigarrow N(m, \sigma^2)$.
Let γ be a vector in $\mathbb{R}^n$. The characteristic function of $\log S_{t+h}$ given $\log S_t$ and Γ_t is defined by :

$$\Psi_{\log S_t}(\gamma, h) = \mathbb{E}\{e^{(\varsigma\gamma)'\log S_{t+h}}/logS_t, \Gamma_t\} \text{ where } t, h \geq 0 \text{ and } \varsigma^2 = -1. \quad (3.4)$$

Using the Feynmann-Kac argument to the model and assuming $g = 0$ and $f = \Psi_{\log S_t}$, we have

$$\frac{\partial \Psi_{\log S_{t^-}}(\gamma, h)}{\partial h} = \mathcal{L}_{\log S,\Gamma}\Psi_{\log S_{t^-}}(\gamma, h) \quad (3.5)$$

where $t, h \geq 0$; $\mathcal{L}_{\log S,\Gamma}$ is the infinitesimal generator of the joint $(\log S_t, \Gamma_t)$ defined by :

Proposition 3.1.

$$\begin{aligned} \mathcal{L}_{\log S,\Gamma} \quad = \quad & tr\left[\left(\nu Q'Q + \frac{\Phi+\Phi'}{2}\Gamma_{t^-} + \Gamma_{t^-}\frac{\Phi+\Phi'}{2}\right)D + 2\Gamma_{t^-}DQ'QD\right] \\ & +\nabla_Y\left(r\check{1} - \frac{1}{2}vec[tr(e_{ii}\Gamma_{t^-})]\right) + \frac{1}{2}\nabla_Y\Gamma_{t^-}\nabla_Y' \\ & +tr(D\sqrt{Q'Q}\rho\nabla_Y\Gamma_{t^-} + \Gamma_{t^-}\nabla_Y'\rho'\sqrt{Q'Q}D) \\ & +\lambda\Psi_{\log S_{t^-}} \times \mathbb{E}\left\{e^{(\varsigma\gamma)'(2\sqrt{\Gamma_{t^-}}J\varphi + JJ'\varphi)} - 1/logS_t, \Gamma_t\right\}, \ with \end{aligned} \quad (3.6)$$

- *$D = (D_{ij})_{1\leq i,j\leq n}$ where $D_{ij} = \frac{\partial}{\partial\Gamma_{ij,t}}$ and $\Gamma_{ij,t}$, $1 \leq i,j \leq n$ are the components of the volatility matrix Γ_t;*

- $\nabla_Y = \left(\frac{\partial}{\partial Y_1}, \cdots, \frac{\partial}{\partial Y_n}\right)'$ *where* $Y_i = \log S_{i,t}$ *is the yield of the i-th underlying in the basket,* $i = 1, ..., n$.

Proof. Let $t, h \geq 0$. The operator $\mathcal{L}_{\log S,\Gamma}$ can be broken down into the following 4 components :

$$\mathcal{L}_{\log S,\Gamma}\Psi_{\log S_{t-}} = \mathcal{L}_{(\log S)^c}\Psi_{\log S_{t-}} + \mathcal{L}_{(\Gamma)^c}\Psi_{\log S_{t-}} + \mathcal{L}_{<(\log S)^c,(\Gamma)^c>}\Psi_{\log S_{t-}} + \mathcal{L}_{jumps} \tag{3.7}$$

with

- $\mathcal{L}_{(\Gamma)^c}$, $\mathcal{L}_{(\log S)^c}$ and $\mathcal{L}_{<(\log S)^c,(\Gamma)^c>}$ are the infinitesimal generators which are demonstrated by Da Fonseca (2007)(see the reference [12]) defined by :

$$\mathcal{L}_{(\log S)^c} = \nabla_Y\left(r\check{1} - \frac{1}{2}vec[tr(e_{ii}\Gamma_{t^-})]\right) + \frac{1}{2}\nabla_Y\Gamma_{t^-}\nabla'_Y; \tag{3.8}$$

$$\mathcal{L}_{(\Gamma)^c} = tr\left[(\nu Q'Q + \Phi\Gamma_{t^-} + \Gamma_{t^-}\Phi')D + 2\Gamma_{t^-}DQ'QD\right]; \tag{3.9}$$

$$\mathcal{L}_{<(\log S)^c,(\Gamma)^c>} = tr(D\sqrt{Q'Q}\rho\nabla_Y\Gamma_{t^-} + \Gamma_{t^-}\nabla'_Y\rho'\sqrt{Q'Q}D); \tag{3.10}$$

- $\mathcal{L}_{jumps}$ is the infinitesimal generator of the jumps defined by :

$$\begin{aligned}\mathcal{L}_{jumps} &= \lambda\mathbb{E}\left\{\Psi(\log S_{t+h} + H) - \Psi(\log S_{t+h})/logS_t, \Gamma_t\right\} \\ &= \lambda\Psi_{\log S_{t-}} \times \mathbb{E}\left\{e^{(\varsigma\gamma)'H} - 1/logS_t, \Gamma_t\right\}\end{aligned} \tag{3.11}$$

where $H = 2\sqrt{\Gamma_{t^-}}J\varphi + JJ'\varphi$.

□

As the yield $\log S_t$ is affine, then we have

$$\Psi_{\log S_t}(\gamma, h) = e^{tr(A(h)\Gamma_t) + B(h)\log S_t + C(h)} \tag{3.12}$$

with $A(h)$, $B(h)$ and $C(h)$ are deterministic functions expressed by:

Proposition 3.2.

$$\begin{aligned}B(h) &= (\varsigma\gamma)', \\ A(h) &= A_{22}(h)^{-1}A_{21}(h), \\ C(h) &= tr\left[r\check{1}h(\varsigma\gamma)' - \frac{\nu}{2}\left(\log A_{22}(h) + h\Upsilon\right)\right] + \\ &\quad \lambda h\left[e^{tr\left[\omega\mu^{-1}\left(\frac{1}{2}(m\tilde{1})^2 + \sqrt{\Gamma_t}(m\tilde{1}) + \frac{1}{2}\sigma^2\Gamma_t\omega\right) - \frac{n}{2}\log\mu\right]} - 1\right].\end{aligned}$$

where $\tilde{1}$ *is a* $n \times n$ *dimensional matrix whose components are equal to* 1 *and*

$$\begin{aligned}\omega &= \varphi(\varsigma\gamma)' + (\varsigma\gamma)\varphi'; \\ \mu &= I_n - \sigma^2 w; \\ \Upsilon &= \frac{(\Phi + (\varsigma\gamma)\rho'\sqrt{Q'Q}) + (\Phi + (\varsigma\gamma)\rho'\sqrt{Q'Q})'}{2}; \\ \begin{bmatrix} A_{11}(h) & A_{12}(h) \\ A_{21}(h) & A_{22}(h)\end{bmatrix} &= \exp\left(h\begin{bmatrix} \Upsilon & -2Q'Q \\ \frac{1}{2}((\varsigma\gamma)(\varsigma\gamma)' - \sum_{j=1}^{n}(\varsigma\gamma_j)e_{jj}) & -\Upsilon\end{bmatrix}\right).\end{aligned}$$

Lemma 3.3. *For any complex matrix $\Omega = (w_{ij})$ such that the eigenvalues are nonzero and $\vartheta \in \mathbb{C}^n$, we have*

$$\int_{\mathbb{R}^n} e^{-x'\Omega x+\vartheta' x}dx = \sqrt{\pi}^n e^{\frac{1}{4}\vartheta'\Omega^{-1}\vartheta-\frac{1}{2}tr(\log\Omega)}. \tag{3.13}$$

Proof. The left integral is equal to

$$e^{\frac{1}{4}\vartheta'\Omega^{-1}\vartheta}\int_{\mathbb{R}^n} e^{-(x-\frac{1}{2}\Omega^{-1}\vartheta)'\Omega(x-\frac{1}{2}\Omega^{-1}\vartheta)}dx \quad = \quad e^{\frac{1}{4}\vartheta'\Omega^{-1}\vartheta}\int_{\mathbb{R}^n} e^{-y'\Omega y}dy, \text{ by imposing}$$
$$y = x - \frac{1}{2}\Omega^{-1}\vartheta. \tag{3.14}$$

We know that the Gaussian integral is given by $\int_{\mathbb{R}^n} e^{-y'y}dy = \sqrt{\pi}^n$ and thus if $a = (a_i)$ is a complex vector where the a_i are non-zero, we have $\int_{\mathbb{R}^n} e^{-\sum\limits_{i=1}^{n} a_i y_i^2}dy = \sqrt{\dfrac{\pi^n}{a_1...a_n}}$ through the change of variable by doing the $x_i = \sqrt{a_i}y_i$. Since any complex matrix is split (see the definition in the reference [30]). Now, let be a complex matrix $\Omega = PDP^{-1}$ where $D = diag(d_i)$ is the diagonal complex matrix where the d_i are non-zero and P its transition complex matrix. We have so

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-y'\Omega y}dy &= \int_{\mathbb{R}^n} e^{-y'Dy}dy \\
&= \sqrt{\frac{\pi^n}{d_1...d_n}} \\
&= \sqrt{\pi^n}e^{-\sum\limits_{i=1}^{n}\log\sqrt{d_i}} \\
&= \sqrt{\pi^n}e^{-\frac{1}{2}tr(\log\Omega)}.
\end{aligned}$$

Hence $3.14 = e^{\frac{1}{4}\vartheta'\Omega^{-1}\vartheta}\sqrt{\pi^n}e^{-\frac{1}{2}tr(\log\Omega)}$. □

Proof of Proposition 3.2. Let $t, h \geq 0$. We have
$\frac{\partial\Psi_{\log S_{t-}}(\gamma,h)}{\partial h} = \left[tr(\frac{\partial A(h)}{\partial h}\Gamma_{t-}) + \frac{\partial B(h)}{\partial h}\log S_{t-} + \frac{\partial C(h)}{\partial h}\right]\Psi_{\log S_{t-}}(\gamma, h)$.
Then, from the expression (3.5), we have

$$\begin{aligned}
&tr(\frac{\partial A(h)}{\partial h}\Gamma_{t-}) + \frac{\partial B(h)}{\partial h}\log S_{t-} + \frac{\partial C(h)}{\partial h} \\
&= B(h)((r1) - \frac{1}{2}vec[tr(e_{ii}\Gamma_{t-})]) + \frac{1}{2}B(h)\Gamma B(h)' + \\
&tr\left[\left(\nu Q'Q + (\frac{\Phi+\Phi'}{2})\Gamma_{t-} + \Gamma_{t-}(\frac{\Phi+\Phi'}{2})\right)A(h) + 2\Gamma_{t-}A(h)Q'QA(h)\right] \\
&+tr[A(h)\sqrt{Q'Q}\rho B(h)\Gamma_{t-} + \Gamma_{t-}B(h)'\rho'\sqrt{Q'Q}A(h)] \\
&+\lambda\mathbb{E}\left[e^{(\varsigma\gamma)'(2\sqrt{\Gamma_{t-}}J\varphi+JJ'\varphi)} - 1/logS_t, \Gamma_t\right]
\end{aligned} \tag{3.15}$$

with the initial conditions $A(0) = 0$, $B(0) = \varsigma\gamma'$ and $C(0) = 0$.
Let us now $\mathbb{E}\left[e^{(\varsigma\gamma)'(2\sqrt{\Gamma_{t-}}J\varphi+JJ'\varphi)}/logS_t, \Gamma_t\right]$.
Let us $\gamma = (\gamma_1, ..., \gamma_n)'$; $\varphi = (\varphi_1, ..., \varphi_n)'$; $\sqrt{\Gamma_t} = (\sigma_{ij,t})_{1\leq i,j\leq n}$ which is symmetrical; $vec(J_k) = (J_{1k}, \cdots, J_{nk})'$ and $vec(\sigma_{k,t}) = (\sigma_{1k,t}, \cdots, \sigma_{nk,t})'$.

So $(\varsigma\gamma)'JJ'\varphi = \sum_{l,j,k=1}^{n}(\varsigma\gamma_j)J_{jk}J_{lk}\varphi_l = \sum_{k=1}^{n}\frac{1}{2}vec(J_k)'(\varphi(\varsigma\gamma)'+(\varsigma\gamma)\varphi')vec(J_k)$ and
$(\varsigma\gamma)'J\sqrt{\Gamma_t}\varphi = \sum_{l,j,k=1}^{n}(\varsigma\gamma_j)J_{jk}\sigma_{lk,t}\varphi_l = \sum_{k=1}^{n}\frac{1}{2}vec(\sigma_{k,t})'(\varphi(\varsigma\gamma)'+(\varsigma\gamma)\varphi')vec(J_k)$.
Let us also $\omega = \varphi(\varsigma\gamma)'+(\varsigma\gamma)\varphi'$ and $\mu = I_n - \sigma^2 w$.
As $vec(J_k) \rightsquigarrow N_n((m\check{1}),\sigma^2 I_n)$, we have

$$\mathbb{E}\left\{e^{(\varsigma\gamma)'(2\sqrt{\Gamma_{t-}}J\varphi+JJ'\varphi)}/logS_t,\Gamma_t\right\}$$
$$= \mathbb{E}\left\{e^{\sum_{k=1}^{n}(\frac{1}{2}vec(J_k))'\omega vec(J_k)+\sum_{k=1}^{n}vec(\sigma_{k,t})'\omega vec(J_k)}/logS_t,\Gamma_t\right\}$$
$$= \prod_{k=1}^{n}\mathbb{E}\left\{e^{\frac{1}{2}vec(J_k)'\omega vec(J_k)+vec(\sigma_{k,t})'\omega vec(J_k)}/logS_t,\Gamma_t\right\} \text{ because } J_{lk} \text{ i.i.d}$$
$$= \prod_{k=1}^{n}e^{\frac{1}{2}(m\check{1})'\omega(m\check{1})+(vec(\sigma_{k,t}))'\omega(m\check{1})}\mathbb{E}\left\{e^{\frac{1}{2}\xi'\sigma^2\omega\xi+\left[(m\check{1})'\omega\sigma+((vec(\sigma_{k,t}))'\omega\sigma)\right]\xi}/logS_t,\Gamma_t\right\}$$

where ξ is a Gaussian random variable in $\mathbb{R}^n$ of density $\dfrac{1}{\sqrt{2}^n\sqrt{\pi}^n}\exp(-\dfrac{1}{2}\varepsilon'\varepsilon)$

$$= \prod_{k=1}^{n}e^{\frac{1}{2}(m\check{1})'\omega(m\check{1})+(vec(\sigma_{k,t}))'\omega(m\check{1})}\int_{\mathbb{R}^n}\frac{e^{-\frac{1}{2}\varepsilon'\mu\varepsilon+\left[(m1)'\omega\sigma+(vec(\sigma_k))'\omega\sigma\right]\varepsilon}}{\sqrt{2}^n\sqrt{\pi}^n}d\varepsilon \quad (3.16)$$

where $\varepsilon = (\varepsilon_1,...,\varepsilon_n)' \in \mathbb{R}^n$ and $d\varepsilon = d\varepsilon_1...d\varepsilon_n$.
As $\parallel \sigma^2\omega \parallel < 1$, so using the Lemma 3.3, we have

$$\begin{aligned}(3.16) \quad = \quad & \prod_{k=1}^{n}e^{\frac{1}{2}(m\check{1})'\omega(m\check{1})+(vec(\sigma_k))'\omega(m\check{1})-\frac{1}{2}tr\log\mu} \\ & e^{\left(\frac{1}{2}(m\check{1})'\omega\sigma+\frac{1}{2}(vec(\sigma_k))'\omega\sigma\right)\left(2\mu^{-1}\right)\left(\frac{1}{2}\sigma\omega(m\check{1})+\frac{1}{2}\sigma\omega vec(\sigma_k)\right)}. \end{aligned} \quad (3.17)$$

Well

$$\frac{1}{2}(m\check{1})'\omega(m\check{1})+\frac{1}{2}(m\check{1})'\omega\sigma\mu^{-1}\sigma\omega(m\check{1}) = \frac{1}{2}(m\check{1})'\omega\left[I_n+\sigma^2\omega\mu^{-1}\right](m\check{1})$$
$$= \frac{1}{2}(m\check{1})'\omega\mu^{-1}(m\check{1}).$$

After some computations, we have
$(vec(\sigma_k))'\omega(m\check{1})+vec(\sigma_k)'\omega\sigma\mu^{-1}\sigma\omega(m\check{1}) = vec(\sigma_k)'\omega\mu^{-1}(m\check{1})$.
Thus

$$\begin{aligned}(3.17) \quad &= \prod_{k=1}^{n}e^{\frac{1}{2}(m\check{1})'\omega\mu^{-1}(m\check{1})+vec(\sigma_k)'\omega\mu^{-1}(m\check{1})+\frac{1}{2}vec(\sigma_k)'\omega\sigma\mu^{-1}\sigma\omega vec(\sigma_k)-\frac{1}{2}tr\log\mu} \\ &= e^{\sum_{k=1}^{n}\frac{1}{2}(m\check{1})'\omega\mu^{-1}(m\check{1})+vec(\sigma_k)'\omega\mu^{-1}(m\check{1})+\frac{1}{2}vec(\sigma_k)'\omega\sigma\mu^{-1}\sigma\omega vec(\sigma_k)-\frac{1}{2}tr\log\mu} \\ &= e^{tr\left[\omega\mu^{-1}\left(\frac{1}{2}(m\tilde{1})^2+\sqrt{\Gamma_t}(m\tilde{1})+\frac{1}{2}\sigma^2\Gamma_t\omega\right)\right]-\frac{n}{2}tr\log\mu}.\end{aligned}$$

So

$$
\begin{aligned}
(3.15) \quad = \quad & B(h)\left((r\check{1}) - \frac{1}{2}vec[tr(e_{ii}\Gamma_{t-})]\right) + \frac{1}{2}B(h)\Gamma B(h)'+ \\
& tr\left[\left(\nu Q'Q + \frac{\Phi+\Phi'}{2}\Gamma_{t-} + \Gamma_{t-}\frac{\Phi+\Phi'}{2}\right)A(h) + 2\Gamma_{t-}A(h)Q'QA(h)\right] \\
& + tr\left[A(h)\sqrt{Q'Q}\rho B(h)\Gamma_{t-} + \Gamma_{t-}B(h)'\rho'\sqrt{Q'Q}A(h)\right] \\
& + \lambda\left[e^{tr\left[\omega\mu^{-1}\left(\frac{1}{2}(m\tilde{1})^2 + \sqrt{\Gamma_t}(m\tilde{1}) + \frac{1}{2}\sigma^2\Gamma_t\omega\right)\right] - \frac{n}{2}tr\log\mu} - 1\right].
\end{aligned}
$$

By identifying the coefficient of $\log S_{t-}$, we have $\frac{\partial B(h)}{\partial h} = 0$ which follows that $B(h) = B(0) = (\varsigma\gamma)'$ for all $h \geq 0$.
Identifying the coefficient of Γ_{t-}, we have

$$
\begin{aligned}
\frac{\partial A(h)}{\partial h} &= -\frac{1}{2}\sum_{i=1}^{n}(\varsigma\gamma_i)e_{ii} + \frac{1}{2}(\varsigma\gamma)(\varsigma\gamma)' + \frac{\Phi+\Phi'}{2}A(h) + A(h)\frac{\Phi+\Phi'}{2} + 2A(h) \\
& \quad Q'QA(h) + A(h)\sqrt{Q'Q}\rho(\varsigma\gamma)' + (\varsigma\gamma)\rho'\sqrt{Q'Q}A(h) \\
&= -\frac{1}{2}\sum_{i=1}^{n}(\varsigma\gamma_i)e_{ii} + \frac{1}{2}(\varsigma\gamma)(\varsigma\gamma)' + \left(\frac{\Phi+\Phi'}{2} + (\varsigma\gamma)\rho'\sqrt{Q'Q}\right)A(h)+ \\
& \quad A(h)\left(\frac{\Phi+\Phi'}{2} + \sqrt{Q'Q}\rho(\varsigma\gamma)'\right) + 2A(h)Q'QA(h) \\
&= -\frac{1}{2}\sum_{i=1}^{n}\varsigma\gamma_i e_{ii} + \frac{1}{2}(\varsigma\gamma)(\varsigma\gamma)' + \Upsilon A(h) + A(h)\Upsilon + 2A(h)Q'QA(h)
\end{aligned}
\tag{3.18}
$$

in the trace operator where $\Upsilon = \frac{(\Phi+(\varsigma\gamma)\rho'\sqrt{Q'Q}) + (\Phi+(\varsigma\gamma)\rho'\sqrt{Q'Q})'}{2}$.
Let us

$$
A(h) = F(h)^{-1}G(h) \text{ with } F(h) \in GL_n(\mathbb{R}) \text{ and } G(h) \in \mathcal{M}_n(\mathbb{R}). \tag{3.19}
$$

We have $0 = A(0) = F(0)^{-1}G(0)$. In this case, we take $G(0) = 0$ and $F(0) = I_n$.
Well, we have $\frac{\partial[F(h)A(h)]}{\partial h} = \frac{\partial F(h)}{\partial h}A(h) + F(h)\frac{\partial A(h)}{\partial h}$. Then, we have

$$
\begin{aligned}
\frac{\partial G(h)}{\partial h} - \frac{\partial F(h)}{\partial h}A(h) &= F(h)\frac{\partial A(h)}{\partial h} \\
&= F(h)\left(-\frac{1}{2}\sum_{i=1}^{n}(\varsigma\gamma_i)e_{ii} + \frac{1}{2}(\varsigma\gamma)(\varsigma\gamma)'\right) + G(h)\Upsilon \\
& \quad + F(h)\Upsilon A(h) + G(h)(2Q'Q)A(h),
\end{aligned}
$$

through (3.18) and (3.19).

Thus

$$
\begin{cases}
\frac{\partial G(h)}{\partial h} = G(h)\Upsilon + F(h)\left(-\frac{1}{2}\sum_{i=1}^{n}(\varsigma\gamma_i)e_{ii} + \frac{1}{2}(\varsigma\gamma)(\varsigma\gamma)'\right) \\
\frac{\partial F(h)}{\partial h} = -2G(h)Q'Q - F(h)\Upsilon.
\end{cases}
$$

So

$$\frac{\partial\left[G(h) \quad F(h)\right]}{\partial h} = \left[G(h) \quad F(h)\right]\begin{bmatrix} \Upsilon & -2Q'Q \\ -\frac{1}{2}\sum_{i=1}^{n}(\varsigma\gamma_i)e_{ii}+\frac{1}{2}(\varsigma\gamma)(\varsigma\gamma)' & -\Upsilon \end{bmatrix}.$$

Then

$$\left[G(h) \quad F(h)\right] = \left[G(0) \quad F(0)\right]\exp\left(h\begin{bmatrix} \Upsilon & -2Q'Q \\ -\frac{1}{2}\sum_{i=1}^{n}(\varsigma\gamma_i)e_{ii}+\frac{1}{2}(\varsigma\gamma)(\varsigma\gamma)' & -\Upsilon \end{bmatrix}\right).$$

Let us $\begin{bmatrix} A_{11}(h) & A_{12}(h) \\ A_{21}(h) & A_{22}(h) \end{bmatrix} = \exp\left(h\begin{bmatrix} \Upsilon & -2Q'Q \\ -\frac{1}{2}\sum_{i=1}^{n}(\varsigma\gamma_i)e_{ii}+\frac{1}{2}(\varsigma\gamma)(\varsigma\gamma)' & -\Upsilon \end{bmatrix}\right).$

We have $G(h) = G(0)A_{11}(h) + F(0)A_{21}(h)$ and $F(h) = G(0)A_{12}(h) + F(0)A_{22}(h)$. As $A(h) = F(h)^{-1}G(h)$, we get

$$A(h) = (G(0)A_{12}(h) + F(0)A_{22}(h))^{-1}(G(0)A_{11}(h) + F(0)A_{21}(h))$$

.

Using the initial conditions $G(0) = 0$ and $F(0) = I_n$, we have

$$A(h) = (A_{22}(h))^{-1}A_{21}(h). \tag{3.20}$$

And finally, by identification

$$\begin{aligned} \frac{\partial C(h)}{\partial h} &= tr[(r\check{1})\gamma' + \nu Q'QA(h)] + \\ &\lambda\left[e^{tr\left[\omega\mu^{-1}\left(\frac{1}{2}(m\tilde{1})^2+\sqrt{\Gamma_t}(m\tilde{1})+\frac{1}{2}\sigma^2\Gamma_t\omega\right)-\frac{n}{2}\log\mu\right]}-1\right] \end{aligned} \tag{3.21}$$

As $\frac{\partial F(h)}{\partial h} = -2G(h)Q'Q - F(h)\Upsilon$, we have $G(h) = -\frac{1}{2}\left[\frac{\partial F(h)}{\partial h} + F(h)\Upsilon\right](Q'Q)^{-1}$. Thus

$$\begin{aligned} tr(\nu Q'QA(h)) &= tr(\nu Q'QF(h)^{-1}G(h)) \\ &= tr\left[\frac{-\nu}{2}F(h)^{-1}\frac{\partial F(h)}{\partial h} - \frac{\nu}{2}\Upsilon\right]. \end{aligned}$$

Then

$$\begin{aligned} (3.21) &= tr\left[(r\check{1})\gamma' + \frac{-\nu}{2}F(h)^{-1}\frac{\partial F(h)}{\partial h} - \frac{\nu}{2}\Upsilon\right] + \\ &\lambda\left[e^{tr\left[\omega\mu^{-1}\left(\frac{1}{2}(m\tilde{1})^2+\sqrt{\Gamma_t}(m\tilde{1})+\frac{1}{2}\sigma^2\Gamma_t\omega\right)-\frac{n}{2}\log\mu\right]}-1\right]. \end{aligned}$$

And thus

$$\begin{aligned} C(h) &= tr\left[(r\check{1})h\gamma' + \frac{-\nu}{2}(\log F(h) - \log F(0)) - \frac{\nu h}{2}\Upsilon\right] \\ &+\lambda h\left[e^{tr\left[\omega\mu^{-1}\left(\frac{1}{2}(m\tilde{1})^2+\sqrt{\Gamma_t}(m\tilde{1})+\frac{1}{2}\sigma^2\Gamma_t\omega\right)-\frac{n}{2}\log\mu\right]}-1\right] \\ &= tr\left[(r\check{1})h\gamma' - \frac{\nu}{2}\left(\log A_{22}(h) + h\Upsilon\right)\right] + \\ &\lambda h\left[e^{tr\left[\omega\mu^{-1}\left(\frac{1}{2}(m\tilde{1})^2+\sqrt{\Gamma_t}(m\tilde{1})+\frac{1}{2}\sigma^2\Gamma_t\omega\right)-\frac{n}{2}\log\mu\right]}-1\right]. \end{aligned}$$

□

3.2. **Characteristic function of Volatility.** Let Γ_t be a solution of the SDE (2.5) with $\nu \geq n$ and Λ be a $n \times n$ dimensional matrix. The characteristic function of Γ_{t+h} given Γ_t is defined by :

$$\Psi_{\Gamma_t}(\Lambda, h) = \mathbb{E}\left\{e^{tr((\varsigma\Lambda)\Gamma_{t+h})}/\Gamma_t\right\} \text{ where } t, h \geq 0. \tag{3.22}$$

Since Γ_t is an affine function then

$$\Psi_{\Gamma}(\Lambda, h) = e^{tr(B(h)\Gamma_t)+c(h)} \tag{3.23}$$

with $B(h)$ and $c(h)$ are deterministic functions expressed by :

Proposition 3.4.

$$\begin{aligned} B(h) &= ((\varsigma\Lambda)B_{12}(h) + B_{22}(h))^{-1}((\varsigma\Lambda)B_{11}(h) + B_{21}(h)), && (3.24)\\ c(h) &= tr\left[-\frac{\nu}{2}\left(\log((\varsigma\Lambda)B_{12}(h) + B_{22}(h)) + h\frac{\Phi + \Phi'}{2}\right)\right] - \lambda h + \\ &\quad \lambda \int_0^h e^{tr\left[B(u)(I_n - 2\sigma^2 B(u))^{-1}\left((m\tilde{1})^2 + 2\sqrt{\Gamma_{t^-}}(m\tilde{1}) + 2\Gamma_{t^-}B(u)\sigma^2\right) - \frac{n}{2}\log \Delta(u)\right]} du && (3.25) \end{aligned}$$

with $\tilde{1}$ is a $n \times n$ dimensional matrix whose components are equal to 1, $\Delta(u) = I_n - 2\sigma^2 B(u)$ and

$$\begin{bmatrix} B_{11}(h) & B_{12}(h) \\ B_{21}(h) & B_{22}(h) \end{bmatrix} = \exp\left(h\begin{bmatrix} \frac{\Phi+\Phi'}{2} & -2Q'Q \\ 0 & -\frac{\Phi+\Phi'}{2} \end{bmatrix}\right). \tag{3.26}$$

Proof. Let be $t, h \geq 0$.
By using the Feynmann-Kac argument on the SDE of Γ_t and imposing $g = 0$ and $f = \Psi_{\Gamma_t}$, we get

$$\frac{\partial \Psi_{\Gamma_{t^-}}(\Lambda, h)}{\partial h} = \mathcal{L}_\Gamma \Psi_{\Gamma_{t^-}}(\Lambda, h), \text{ with} \tag{3.27}$$

$$\begin{aligned} \mathcal{L}_\Gamma &= tr\left[\left(\nu Q'Q + \frac{\Phi + \Phi'}{2}\Gamma_{t^-} + \Gamma_{t^-}\frac{\Phi + \Phi'}{2}\right)D + 2\Gamma_{t^-}DQ'QD\right] \\ &\quad + \lambda\Psi_{\Gamma_{t^-}}\mathbb{E}\left\{e^{tr\left(B(h)(2\sqrt{\Gamma_{t^-}}J + JJ'\right)} - 1/\Gamma_t\right\} \end{aligned}$$

where $D = (D_{ij})_{ij}$ and $D_{ij} = \frac{\partial}{\partial \Gamma_{ij,t}}$. We have also
$\frac{\partial \Psi_{\log S_{t^-}}(\gamma, h)}{\partial h} = \left[tr\left(\frac{\partial B(h)}{\partial h}\Gamma_{t^-}\right) + \frac{\partial c(h)}{\partial h}\right]\Psi_{\Gamma_{t^-}}(\Lambda, h)$.
So, from the expression (3.27), we have

$$\begin{aligned} & tr\left(\frac{\partial B(h)}{\partial h}\Gamma_{t^-}\right) + \frac{\partial c(h)}{\partial h} \\ &= tr\left[\left(\nu QQ' + \frac{\Phi + \Phi'}{2}\Gamma_{t^-} + \Gamma_{t^-}\frac{\Phi + \Phi'}{2}\right)B(h) + 2\Gamma_{t^-}B(h)Q'QB(h)\right] \\ &+ \lambda\mathbb{E}\left\{e^{tr\left[B(h)\left(2\sqrt{\Gamma_{t^-}}J + JJ'\right)\right]} - 1/\Gamma_t\right\} \end{aligned} \tag{3.28}$$

with the initial conditions $B(0) = \varsigma\Lambda$ and $c(0) = 0$.
Let's first determine $\mathbb{E}\left\{e^{tr\left[B(h)\left(2\sqrt{\Gamma_{t^-}}J + JJ'\right)\right]}/\Gamma_t\right\}$:

using $J = (J_{lk})_{lk}$ where J_{lk} are the i.i.d normal random variables with $J_{lk} \rightsquigarrow N(m, \sigma^2)$ and $\sqrt{\Gamma_t} = (\sigma_{ij})_{ij}$ is a symmetrical matrix, we have

$$\begin{aligned} tr(B(h)JJ') &= \sum_{l,k,p=1}^{n} J_{lp}B_{lk}(h)J_{kp} \\ &= \sum_{p=1}^{n}(vec(J_p))'B(h)vec(J_p) \end{aligned}$$

and

$$\begin{aligned} tr(B(h)\sqrt{\Gamma}J &= \sum_{l,k,p=1}^{n} B_{lk}(h)\sigma_{lp}J_{pk} \\ &= \sum_{p=1}^{n}(vec(\sigma_p))'B(h)vec(J_p). \end{aligned}$$

Hence, we have

$$\begin{aligned} &\mathbb{E}\left[e^{tr\left[B(h)\left(2\sqrt{\Gamma_{t^-}}J+JJ'\right)\right]}/\Gamma_t\right] \\ &= \mathbb{E}\left[e^{\sum\limits_{p=1}^{n}(vec(J_p))'B(h)vec(J_p)+2(vec(\sigma_p))'B(h)vec(J_p)}/\Gamma_t\right] \\ &= \prod_{p=1}^{n}\mathbb{E}\left[e^{vec(J_p)'B(h)vec(J_p)+2(vec(\sigma_p))'B(h)vec(J_p)}/\Gamma_t\right] \text{ because } (J_{lk}) \text{ i.i.d} \\ &= \prod_{k=1}^{n} e^{(m\check{1})'B(h)m\check{1}+2vec(\sigma_k)'B(h)m\check{1}}\mathbb{E}\left[e^{\xi'\sigma^2B(h)\xi+\left(2(m\check{1})'B(h)\sigma)+2vec(\sigma_k)'B(h)\sigma\right)\xi}/\Gamma_t\right] \\ &= \prod_{k=1}^{n} e^{(m\check{1})'B(h)(m\check{1})+2vec(\sigma_k)'B(h)(m\check{1})}\int\frac{e^{-\varepsilon'\frac{1}{2}\Delta(h)\varepsilon+\left(2(m\check{1})'B(h)\sigma+2vec(\sigma_k)'B(h)\sigma\right)\varepsilon}}{\sqrt{2}^n\sqrt{\pi}^n}d\varepsilon \\ &= \prod_{k=1}^{n} e^{\left((m\check{1})'B(h)\sigma+(vec(\sigma_k))'B(h)\sigma\right)\left(2\Delta(h)^{-1}\right)\left(\sigma B(h)(m\check{1})+\sigma B(h)vec(\sigma_k)\right)} \\ &e^{(m\check{1})'B(h)m\check{1}+2(vec(\sigma_k))'B(h)(m\check{1})-\frac{1}{2}\log\Delta(h)} \text{through the Lemma 3.3 where} \\ &\Delta(h) = I_n - 2\sigma^2B(h) \\ &= \prod_{k=1}^{n} e^{tr\left[B(h)\Delta(h)^{-1}[(m\check{1})(m\check{1})'+2(m\check{1})(vec(\sigma_k))'+2\sigma^2B(h)vec(\sigma_k)(vec(\sigma_k))']-\frac{1}{2}\log\Delta(h)\right]} \\ &= e^{tr\left[B(h)\Delta(h)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_{t^-}}(m\tilde{1})+2\Gamma_{t^-}B(h)\sigma^2\right)-\frac{n}{2}\log\Delta(h)\right]}. \end{aligned}$$

Thus

$$\begin{aligned} (3.28) \quad = \quad & tr\left[\left(\nu QQ' + \frac{\Phi+\Phi'}{2}\Gamma_{t^-} + \Gamma_{t^-}\frac{\Phi+\Phi'}{2}\right)B(h) + 2\Gamma_{t^-}B(h)Q'QB(h)\right] + \\ & \lambda\left[e^{tr\left[B(h)\Delta(h)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_{t^-}}(m\tilde{1})+2\Gamma_{t^-}B(h)\sigma^2\right)-\frac{n}{2}\log\Delta(h)\right]} - 1\right]. \end{aligned}$$

Identifying the coefficient of Γ_{t^-}, we get

$$\frac{\partial B(h)}{\partial h} = (\frac{\Phi+\Phi'}{2})B(h) + B(h)(\frac{\Phi+\Phi'}{2}) + 2B(h)Q'QB(h).$$

Let us $B(h)=F(h)^{-1}G(h)$ with $F(h)\in GL_n(\mathbb{R})$ and $G(h)\in\mathcal{M}_n(\mathbb{R})$. We have $\varsigma\Lambda=B(0)=F(0)^{-1}G(0)$. In this case, take $G(0)=\varsigma\Lambda$ and $F(0)=I_n$. In addition, we have $\frac{\partial[F(h)B(h)]}{\partial h}=\frac{\partial F(h)}{\partial h}B(h)+F(h)\frac{\partial B(h)}{\partial h}$. Then,

$$\begin{aligned}\frac{\partial G(h)}{\partial h}-\frac{\partial F(h)}{\partial h}B(h) &= F(h)\frac{\partial B(h)}{\partial h}\\ &= F(h)\left[\frac{\Phi+\Phi'}{2}B(h)+B(h)\frac{\Phi+\Phi'}{2}+2B(h)Q'QB(h)\right]\\ &= G(h)\frac{\Phi+\Phi'}{2}+F(h)\frac{\Phi+\Phi'}{2}B(h)+2G(h)Q'QB(h).\end{aligned}$$

Hence, $\begin{cases}\frac{\partial G(h)}{\partial h}=G(h)(\frac{\Phi+\Phi'}{2})\\ \frac{\partial F(h)}{\partial h}=-2G(h)Q'Q-F(h)(\frac{\Phi+\Phi'}{2}).\end{cases}$

Writing in matrix form, we have $\frac{\partial\begin{bmatrix}G(h) & F(h)\end{bmatrix}}{\partial h}=\begin{bmatrix}G(h) & F(h)\end{bmatrix}\begin{bmatrix}\frac{\Phi+\Phi'}{2} & -2Q'Q\\ 0 & -\frac{\Phi+\Phi'}{2}\end{bmatrix}$.

So, the solution is $\begin{bmatrix}G(h) & F(h)\end{bmatrix}=\begin{bmatrix}G(0) & F(0)\end{bmatrix}\exp\left(h\begin{bmatrix}\frac{\Phi+\Phi'}{2} & -2Q'Q\\ 0 & -\frac{\Phi+\Phi'}{2}\end{bmatrix}\right)$.

Let us $\begin{bmatrix}B_{11}(h) & B_{12}(h)\\ B_{21}(h) & B_{22}(h)\end{bmatrix}=\exp\left(h\begin{bmatrix}\frac{\Phi+\Phi'}{2} & -2Q'Q\\ 0 & -\frac{\Phi+\Phi'}{2}\end{bmatrix}\right)$.

We have $G(h)=G(0)B_{11}(h)+F(0)B_{21}(h)$ and $F(h)=G(0)B_{12}(h)+F(0)B_{22}(h)$. Then, as $B(h)=F(h)^{-1}G(h)$, we have

$$B(h)=(G(0)B_{12}(h)+F(0)B_{22}(h))^{-1}(G(0)B_{11}(h)+F(0)B_{21}(h)).$$

Moreover, as $G(0)=\varsigma\Lambda$ and $F(0)=I_n$, then we have

$$B(h)=((\varsigma\Lambda)B_{12}(h)+B_{22}(h))^{-1}((\varsigma\Lambda)B_{11}(h)+B_{21}(h)). \tag{3.29}$$

Finally, by identification we have

$$\begin{aligned}\frac{\partial c(h)}{\partial h} &= tr[\nu QQ'B(h)]-\lambda+\\ &\quad \lambda\left[e^{tr\left[B(h)\Delta(h)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_{t^-}}(m\tilde{1})+2\Gamma_{t^-}B(h)\sigma^2\right)-\frac{n}{2}\log\Delta(h)\right]}\right]\end{aligned} \tag{3.30}$$

with $c(0)=0$. Since $\frac{\partial F(h)}{\partial h}=-2G(h)Q'Q-F(h)(\frac{\Phi+\Phi'}{2})$, we get

$$G(h)=-\frac{1}{2}\left[\frac{\partial F(h)}{\partial h}+F(h)(\frac{\Phi+\Phi'}{2})\right](Q'Q)^{-1}.$$

Thus, $tr(\nu Q'QB(h))=tr\left[(\nu QQ'F(h)^{-1}G(h)\right].=tr\left[\frac{-\nu}{2}F(h)^{-1}\frac{\partial F(h)}{\partial h}-\frac{\nu}{2}(\frac{\Phi+\Phi'}{2})\right]$.
So,

$$\begin{aligned}(3.30) &= tr\left[\frac{-\nu}{2}F(h)^{-1}\frac{\partial F(h)}{\partial h}-\frac{\nu}{2}\frac{\Phi+\Phi'}{2}\right]-\lambda\\ &\quad +\lambda e^{tr\left[B(h)\Delta(h)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_{t^-}}(m\tilde{1})+2\Gamma_{t^-}B(h)\sigma^2\right)-\frac{n}{2}\log\Delta(h)\right]}.\end{aligned}$$

And thus

$$\begin{aligned} c(h) &= tr\left[\frac{-\nu}{2}(\log F(h) - \log F(0)) - \frac{\nu h}{2}\frac{\Phi+\Phi'}{2}\right] - \lambda h \\ &\quad + \lambda \int_0^h e^{tr\left[B(u)\Delta(u)^{-1}\left((m\tilde{1})^2 + 2\sqrt{\Gamma_{t^-}}(m\tilde{1}) + 2\Gamma_{t^-}B(u)\sigma^2\right) - \frac{n}{2}\log\Delta(u)\right]} du \\ &= tr\left[-\frac{\nu}{2}\left(\log((\varsigma\Lambda)B_{12}(h) + B_{22}(h)) + h\frac{\Phi+\Phi'}{2}\right)\right] - \lambda h \\ &\quad + \lambda \int_0^h e^{tr\left[B(u)\Delta(u)^{-1}\left((m\tilde{1})^2 + 2\sqrt{\Gamma_{t^-}}(m\tilde{1}) + 2\Gamma_{t^-}B(u)\sigma^2\right) - \frac{n}{2}\log\Delta(u)\right]} du \end{aligned}$$

where $\Delta(u) = I_n - 2\sigma^2 B(u)$ for all $u \in [0; h]$.

□

Theorem 3.5. *Let $h \geq 0$. If Φ is an invertible matrix and $\left[I_n - 2(\varsigma\Lambda)Q'Q(\Phi+\Phi')^{-1}(e^{(\Phi+\Phi')h} - I_n)\right]^{-1}$ exists in the trace operator, we have*

$$B(h) = \left[I_n - 2(\varsigma\Lambda)Q'Q(\Phi+\Phi')^{-1}(e^{(\Phi+\Phi')h} - I_n)\right]^{-1}(\varsigma\Lambda)e^{(\Phi+\Phi')h}, \tag{3.31}$$

$$\begin{bmatrix} B_{11}(h) & B_{12}(h) \\ B_{21}(h) & B_{22}(h) \end{bmatrix} = \begin{bmatrix} e^{(\frac{\Phi+\Phi'}{2})h} & -2(\Phi+\Phi')^{-1}(e^{(\frac{\Phi+\Phi'}{2})h} - e^{-(\frac{\Phi+\Phi'}{2})h})Q'Q \\ 0 & e^{-(\frac{\Phi+\Phi'}{2})h} \end{bmatrix}. \tag{3.32}$$

Proof. Let be $h \geq 0$ and $T = \begin{bmatrix} \frac{\Phi+\Phi'}{2} & -2Q'Q \\ 0 & -\frac{\Phi+\Phi'}{2} \end{bmatrix}$, $T^s = (T_{ij}^{(s)})_{ij}$, $s \in \mathbb{N}$.

In the trace operator, we have $T_{ij}^{(0)} = I_n$ si $i = j$ and 0 otherwise; $T_{11}^{(1)} = \frac{\Phi+\Phi'}{2}$; $T_{12}^{(1)} = -2Q'Q$; $T_{21}^{(1)} = 0$; $T_{22}^{(1)} = -(\frac{\Phi+\Phi'}{2})$; $T_{11}^{(2)} = (\frac{\Phi+\Phi'}{2})^2$; $T_{12}^{(2)} = -2(\frac{\Phi+\Phi'}{2})Q'Q + 2Q'Q(\frac{\Phi+\Phi'}{2}) = 0$; $T_{21}^{(2)} = 0$; $T_{22}^{(2)} = (\frac{\Phi+\Phi'}{2})^2$.

Now, let us consider $p \geq 1$, in the trace operator, reasoning by recurrence, we have $T_{11}^{2(p+1)} = T_{11}^{(2)}T_{11}^{(2p)} + T_{12}^{(2)}T_{21}^{(2p)} = (\frac{\Phi+\Phi'}{2})^{2(p+1)}$; $T_{12}^{2(p+1)} = T_{11}^{(2)}T_{12}^{(2p)} + T_{12}^{(2)}T_{22}^{(2p)} = 0$; $T_{21}^{2(p+1)} = T_{21}^{(2)}T_{11}^{(2p)} + T_{22}^{(2)}T_{21}^{(2p)} = 0$; $T_{22}^{2(p+1)} = T_{21}^{(2)}T_{12}^{(2p)} + T_{22}^{(2)}T_{22}^{(2p)} = (\frac{\Phi+\Phi'}{2})^{2(p+1)}$.

Then using the values $T_{ij}^{(1)}$ and $T_{ij}^{(2p)}$ above, we have, for all $p \geq 1$ $T_{11}^{2p+1} = T_{11}^{(1)}T_{11}^{(2p)} + T_{12}^{(1)}T_{21}^{(2p)} = (\frac{\Phi+\Phi'}{2})^{2p+1}$; $T_{12}^{2p+1} = T_{11}^{(1)}T_{12}^{(2p)} + T_{12}^{(1)}T_{22}^{(2p)} = -2\left(\frac{\Phi+\Phi'}{2}\right)^{2p}Q'Q$; $T_{21}^{2p+1} = T_{21}^{(1)}T_{11}^{(2p)} + T_{22}^{(1)}T_{21}^{(2p)} = 0$; $T_{22}^{2p+1} = T_{21}^{(1)}T_{12}^{(2p)} + T_{22}^{(1)}T_{22}^{(2p)} = -(\frac{\Phi+\Phi'}{2})^{2p+1}$.

Well, we have

$$\begin{aligned} \begin{bmatrix} B_{11}(h) & B_{12}(h) \\ B_{21}(h) & B_{22}(h) \end{bmatrix} &= e^{hT} \\ &= \sum_{s=0}^{+\infty}\frac{(hT)^s}{s!} \\ &= \sum_{p=0}^{+\infty}\frac{(hT)^{(2p)}}{(2p)!} + \sum_{p=0}^{+\infty}\frac{(hT)^{(2p+1)}}{(2p+1)!}. \end{aligned}$$

$$= \begin{bmatrix} e^{(\frac{\Phi+\Phi'}{2})h} & -(\frac{\Phi+\Phi'}{2})^{-1}(e^{(\frac{\Phi+\Phi'}{2})h} - e^{-(\frac{\Phi+\Phi'}{2})h})Q'Q \\ 0 & e^{-(\frac{\Phi+\Phi'}{2})h} \end{bmatrix}.$$

The value of $B(h)$ is obtained through the expression 3.29 and the $B_{ij}(h)$ above. □

Remark. *The values $B(h)$ and $B_{ij}(h)$ are not their true values but when they are in the trace operator, they are reduced to these values. Fortunately, we can use these values because at the beginning they are defined in the trace operator (see characteristic function of volatility).*

4. Characteristic study of the model

4.1. Stationarity of Γ_t.

Theorem 4.1. *Let $h \geq 0$. If the following conditions are satisfied :*

(i) $\nu \geq n$,
(ii) Φ *is a negative definite matrix,*
(iii) λ *is a parameter in function of h such as λh converges when h tends to $+\infty$ or*
(iv) m *and σ are the parameters in function of h such as mh and σh converge when h tends to $+\infty$.*

So the conditional process Γ_{t+h} given Γ_t is stationary and converges to a Wishart distribution.

Let be $A \in \mathcal{M}_n(\mathbb{R})$.

Theorem 4.2. *Let f be an endomorphism of a vector space E. If f is split then there exists a endomorphism single joint (u, v) such as $f = u+v$; u is diagonalizable and v is nilpotent; u commute with v.*

Proof. see the reference [30]. □

Corollary 4.3. *If A is a negative definite matrix and if it is triangularizable then it is invertible and e^{Ah} tends to 0 when h tends to infinity.*

Proof. It is obvious that if A is a triangularizable and negative definite matrix then A is invertible. Moreover, under these conditions, e^{Ah} tends to 0 when h tends to infinity. Indeed, A is similar to a triangular matrix T. So we can find a passage matrix P such as $A = PTP^{-1}$. Moreover, according to Theorem 4.2, we can find a diagonal matrix D composed of the eigenvalues of A which are negative non-zero and a matrix N which is nilpotent and commutes with D such as $T = D + N$.
Let us assume N is a nilpotent matrix of index p.
So, for all $h \geq 0$, we have

$$\begin{aligned} e^{Ah} &= e^{Th} \\ &= e^{(D+N)h} \\ &= e^{Dh}e^{Nh}, \text{ through the Campbell-Backer-Haussdorff formula} \\ &= e^{Dh}\left(I_n + \frac{Nh}{1!} + \frac{N^2h^2}{2!} + ... + \frac{N^{p-1}h^{p-1}}{(p-1)!}\right) \underset{h\longrightarrow+\infty}{\longrightarrow} 0. \end{aligned}$$

□

Lemma 4.4. *If A is a negative definite matrix then $e^{(A+A')h}$ tend to 0 when h tend to the infinite.*

Proof. As A is a negative defined matrix, then A' and $A+A'$ are also negative. The result follows so applying the Corollary 4.3. □

Proof of Theorem 4.1. $\Psi_\Gamma(\Lambda, h)$ admits a limit when h tends to infinity through (3.23) if $B(h)$ and $c(h)$ admit finite limits or tend to $-\infty$. As Φ is a negative definite matrix then $B(h)$ converges to zero when h tends to infinity through the Lemma 4.4.
In addition, we see that the characteristic function our model is similar to WASC model by adding the term :

$$\lambda \int_0^h e^{tr\left[B(u)\Delta(u)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_{t^-}}(m\tilde{1})+2\Gamma_{t^-}B(u)\sigma^2\right)-\frac{n}{2}\log\Delta(u)\right]}-1du \qquad (4.1)$$

in expression of $c(h)$ in equation (3.24). So, in long term (that is, for h large enough), we can regularize with λ or m and σ for the expression (4.1) don't tend to infinity when h tends to infinity under the condition of stationarity of WASC model which are Φ is a negative definite matrix. Using the change of variable by doing $u = vh$, (4.1) becomes

$$\lambda h \int_0^1 e^{tr\left[B(vh)\Delta(vh)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_{t^-}}(m\tilde{1})+2\Gamma_{t^-}B(vh)\sigma^2\right)-\frac{n}{2}\log\Delta(vh)\right]}-1dv. \qquad (4.2)$$

Let us for all $v \in]0,1[$ and $h \geq 0$,
$g(v,h) = \left[e^{tr\left[B(vh)\Delta(vh)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_{t^-}}(m\tilde{1})+2\Gamma_{t^-}B(vh)\sigma^2\right)-\frac{n}{2}\log\Delta(vh)\right]}-1\right]\lambda h$,
$G(u) = \frac{g(v,\frac{u}{v})}{\lambda h}$, $u \in]0,1[$. We have $G(u)$ tends to 0 when h tends to $+\infty$. Thus, we can find $T_1 < u$, we have $\mid G(u) \mid < \frac{1}{2}$.
Let firstly λ be a parameter in function of h such as λh converges when h tends to $+\infty$ where $\dot{\lambda}$ the limit of λh when h tends to $+\infty$. Thus, we can find $T_2 < h$, we have $\lambda h < \frac{1}{2} + \dot{\lambda}$. Distinguish six cases :
Case 1: $0 \leq h \leq 100$ **and** $0 < v < 1$**.** We have $\mid g(v,h) \mid < A_1$ where $A_1 = \underset{\substack{0<v<1\\0\leq h\leq 100}}{sup}\ g(v,h)$.
Case 2: $100 < u = vh \leq T_1 \leq h$ **and** $0 < v < 1$**.** We have $\mid g(v,h) \mid < A_2$ where $A_2 = \underset{100\leq u\leq T_1}{Sup} \mid G(u) \mid \underset{T_1\leq h\leq T_2}{max}(\dot{\lambda}+\frac{1}{2}, \lambda h)$.
Case 3: $100 < T_1 < u < h$ **and** $0 < v < 1$**.** We have $\mid g(v,h) \mid < A_3$ where $A_3 = \frac{1}{2}\underset{T_1\leq h\leq T_2}{max}(\dot{\lambda}+\frac{1}{2}, \lambda h)$.
Case 4: $100 < u < h < T_1$ **and** $0 < v < 1$**.** We have $\mid g(v,h) \mid < A_4$ where $A_4 = \underset{100\leq u\leq T_1}{Sup} \mid G(u) \mid \underset{100\leq h\leq T_1}{max}(\lambda h)$.
Case 5: $0 < u < 100 < T_1 < h$ **and** $0 < v < 1$**.** We have $\mid g(v,h) \mid < A_5$ where $A_5 = \underset{0\leq u\leq 100}{Sup} \mid G(u) \mid \underset{T_1\leq h\leq T_2}{max}(\dot{\lambda}+\frac{1}{2}, \lambda h)$.
Case 6: $0 < u < 100 < h < T_1$ **and** $0 < v < 1$**.** We have $\mid g(v,h) \mid < A_6$ where $A_6 = \underset{0\leq u\leq 100}{Sup} \mid G(u) \mid \underset{100\leq h\leq T_1}{max}(\lambda h)$.
Hence, $\mid g(v,h) \mid < A$ where $A = \max_{i=1,2,3,4,5,6}(A_i)$ for all $v \in]0,1[$ and $h \geq 0$.
In addition, $g(v,h)$ tends to 0 when h tends to $+\infty$ for all $v \in]0,1[$. So, by using

the dominated convergence of Lebesgue theorem (see the reference [7]), (4.2) tends to 0 when h tends to $+\infty$. And thus, $\Psi_{\Gamma_t}(\Lambda, h)$ tends to $e^{-\frac{n}{2}tr\log(I_n-2\Omega_1(\varsigma\Lambda))}$ which is a characteristic function of Wishart distribution with $vec(\Omega_1) = -(I_n\otimes\Phi+\Phi\otimes I_n)^{-1}\circ vec(Q'Q)$ (see the reference [13]).
Now, if m and σ are the parameters in function of h such as mh and σh converge when h tends to $+\infty$. Let $\dot{m}$ be the limit of mh when h tends to $+\infty$. We have $\underset{0\leq u\leq h}{sup}(G(u)\lambda h)$ tends to 0 when h tends to $+\infty$. So, we can find $T < h$, $\mid \underset{0\leq u\leq h}{sup}(G(u)\lambda h)\mid < \frac{1}{2}$.
Distinguish also six cases :
Case 1: $0 \leq h \leq 100$ **and** $0 < v < 1$. We have $\mid g(v,h)\mid < A_1$.
Case 2: $100 < u = vh \leq T \leq h$ **and** $0 < v < 1$. We have $\mid g(v,h)\mid < \frac{1}{2} + \underset{0\leq u}{sup}tr(2B(u)\dot{m}\tilde{1}\sqrt{\Gamma_t})$.
Case 3: $100 < T < u < h$ **and** $0 < v < 1$. We have $\mid g(v,h)\mid < \frac{1}{2}$.
Case 4: $100 < u < h < T$ **and** $0 < v < 1$. We have $\mid g(v,h)\mid < B_1$ where $B_1 = \underset{100\leq u\leq T}{Sup}\mid G(u)\mid \underset{100\leq h\leq T}{max}(\lambda h)$.
Case 5: $0 < u < 100 < T < h$ **and** $0 < v < 1$. We have $\mid g(v,h)\mid < \frac{1}{2}$.
Case 6: $0 < u < 100 < h < T_1$ **and** $0 < v < 1$. We have $\mid g(v,h)\mid < B_2$ where $B_2 = \underset{0\leq u\leq 100}{Sup}\mid G(u)\mid \underset{100\leq h\leq T}{max}(\lambda h)$.
Hence, $\mid g(v,h)\mid < B$ where $B = \max_{i=1,2}(A_1, B_i, \frac{1}{2})$ for all $v\in]0,1[$ and $h \geq 0$.
Since $g(v,h)$ tends to 0 when h tends to $+\infty$ for all $v\in]0,1[$. So using the dominated convergence of Lebesgue theorem, (4.2) tends to 0 when h tends to $+\infty$ and thus $\Psi_{\Gamma_t}(\Lambda,h)$ tends also to $e^{-\frac{n}{2}tr\log(I_n-2\Omega_1(\varsigma\Lambda))}$. □

Theorem 4.5. *Let Γ_∞ be the limit of the stationary distribution of Γ_t. Then, Γ_∞ is the solution of :*

$$\Gamma_\infty\Phi' + \Phi\Gamma_\infty + \lambda(m\tilde{1})\sqrt{\Gamma_\infty} + \lambda\sqrt{\Gamma_\infty}(m\tilde{1}) = -\nu Q'Q - \lambda n\sigma^2 I_n - n\lambda(m\tilde{1})^2. \quad (4.3)$$

Proof. Using the following SDE of Γ_t :
$d\Gamma_t = (\nu Q'Q + \Phi\Gamma_t + \Gamma_t\Phi' dt + \sqrt{\Gamma_t}dW_t\sqrt{Q'Q} + \sqrt{Q'Q}(dW_t)'\sqrt{\Gamma_t} + \sqrt{\Gamma_t}dP_t + (dP)'\sqrt{\Gamma_t} + dP_t(dP_t)'$ with
$P_t = \sum_{j=1}^{N_t} J_j$ where $J_j = (J_{j,kl})_{kl}$ is a $n\times n$ dimensional matrix such as $J_{j,kl}$ for all j, k and l are the i.i.d normal random variables with $J_{j,kl} \rightsquigarrow N(m,\sigma^2)$, we have for a very small positive h,
$\Gamma_{t+h} - \Gamma_t = (\nu Q'Q + \Gamma_t\Phi' + \Phi\Gamma_t)h + \sqrt{\Gamma_t}(W_{t+h} - W_t)'\sqrt{Q'Q} + \sqrt{Q'Q}(W_{t+h} - W_t)\sqrt{\Gamma_t} + \sqrt{\Gamma_t}(P_{t+h} - P_t) + (P_{t+h} - P_t)'\sqrt{\Gamma_t} + (P_{t+h} - P_t)(P_{t+h} - P_t)'$.
Moving to conditional expectation, we have
$\mathbb{E}\{\Gamma_{t+h} - \Gamma_t/\Gamma_t\} = (\nu Q'Q + \Gamma_t\Phi' + \Phi\Gamma_t)h + \sqrt{\Gamma_t}(\mathbb{E}\{W_{t+h} - W_t/\Gamma_t\})'\sqrt{Q'Q} + \sqrt{Q'Q}\mathbb{E}\{W_{t+h} - W_t/\Gamma_t\}\sqrt{\Gamma_t} + (\mathbb{E}\{P_{t+h} - P_t/\Gamma_t\})'\sqrt{\Gamma_t} + \sqrt{\Gamma_t}\mathbb{E}\{P_{t+h} - P_t/\Gamma_t\} + \mathbb{E}\{(P_{t+h} - P_t)(P_{t+h} - P_t)'/\Gamma_t\}$.
Using the independent and stationary increasement of a Brownian motion and the compound Poisson process, we have
$\mathbb{E}\{\Gamma_{t+h}-\Gamma_t/\Gamma_t\} = (\nu Q'Q+\Gamma_t\Phi'+\Phi\Gamma_t)h+\sqrt{\Gamma_t}(\mathbb{E}\{W_h-W_0\})'\sqrt{Q'Q}+\sqrt{Q'Q}\mathbb{E}\{W_h-W_0\}\sqrt{\Gamma_t}+(\mathbb{E}\{P_h-P_0\})'\sqrt{\Gamma_t}+\sqrt{\Gamma_t}\mathbb{E}\{P_h-P_0\}+\mathbb{E}\{(P_h-P_0)(P_h-P_0)'\}$ with $W_0 = 0$ and $P_0 = 0$.

Thus, its latter is equal to
$(\nu Q'Q+\Gamma_t\Phi'+\Phi\Gamma_t)h+\lambda h(m\tilde{1})\sqrt{\Gamma_t}+\lambda h\sqrt{\Gamma_t}(m\tilde{1})+\lambda hn\sigma^2 I_n+n\lambda h(m\tilde{1})^2$.
Doing tend t to $+\infty$, we have
$0=\nu Q'Q+\Gamma_\infty\Phi'+\Phi\Gamma_\infty+\lambda(m\tilde{1})\sqrt{\Gamma_\infty}+\lambda\sqrt{\Gamma_\infty}(m\tilde{1})+\lambda n\sigma^2 I_n+n\lambda(m\tilde{1})^2$.
That is, Γ_∞ is the solution of
$\Gamma_\infty\Phi'+\Phi\Gamma_\infty+\lambda(m\tilde{1})\sqrt{\Gamma_\infty}+\lambda\sqrt{\Gamma_\infty}(m\tilde{1})=-\nu Q'Q-\lambda n\sigma^2 I_n-n\lambda(m\tilde{1})^2$. □

4.2. **Correlation between yield and its volatility.** We assume that the model checks:
i) each component of the vector B_t is independent with the one matrix $\tilde{W}_t$ (see the third equation in (1.3));
ii) the continuous part of the yield $\log S_t$ and the continuous part of its volatility Γ_t are linearly correlated.

Theorem 4.6. *The covariance between the each component of vector yield noise* $d\log S_t$ *and the one volatility noise matrix* $d\Gamma_t$ *is given by for all* $i,j,h=1,...,n$,

$$cov(d(\log S_{h,t})^c\ ,\ d(\Gamma_{ij,t})^c)=\left(\Gamma_{hi,t}\sum_{l=1}^n Q_{lj}\rho_l+\Gamma_{hj,t}\sum_{l=1}^n Q_{li}\rho_l\right)dt,\ with \tag{4.4}$$

- $\log S_{.,t}$ *is the component of the yield vector* $\log S_t$*,*
- $\rho_.$ *is the component of vector* ρ*,*
- $\Gamma_{..,t}$ *is the component of the volatility matrix* Γ_t *and*
- $Q_{..}$ *is the component of the matrix* $Q'Q$*.*

Proof. From the expressions $\sqrt{\Gamma_t}=(\sigma_{ij,t})_{1\le i,j\le n}$ which is symmetrical and $\Gamma_t=(\Gamma_{ij,t})_{i,j=1,...,n}$, we get

$$\Gamma_{ij,t}=\sum_{l=1}^n \sigma_{il,t}\sigma_{jl,t}. \tag{4.5}$$

Now, let be $i,j,h\in\{1,...,n\}$.
We have $cov(d(\log S_{h,t})^c\ ,\ d(\Gamma_{ij,t})^c)=<d(\log S_{h,t})^c\ ,\ d(\Gamma_{ij,t})^c>$ with $d(\log S_{h,t})^c$ is the yield noise of $\log S_{h,t}$ in the continuous part which is the h-th line of $d\log S_t$ defined in equation (1.3) by $d(\log S_{h,t})^c=\left(\mu_h-\dfrac{\Gamma_{hh,t}}{2}\right)dt+\sum\limits_{k=1}^n \sigma_{hk,t}dZ_{k,t}$. And $d(\Gamma_{ij,t})^c$ is the component i-th row and j-th column of $d(\Gamma_t)^c$ with

$$\begin{aligned} d(\Gamma_{ij,t})^c &= \left(\nu\sum_{l=1}^n Q_{il}Q_{jl}+\sum_{l=1}^n \Phi_{il}\Gamma_{lj,t}+\sum_{l=1}^n \Gamma_{il,t}\Phi_{jl}\right)dt \\ &\quad+\sum_{m,l=1}^n(\sigma_{im,t}dW_{ml,t}Q_{lj}+\sigma_{jm,t}dW_{ml,t}Q_{li}). \end{aligned} \tag{4.6}$$

So

$$\begin{aligned} &<d(\log S_{h,t})^c\ ,\ d(\Gamma_{ij,t})^c> \\ &=<\sum_{k=1}^n \sigma_{hk,t}dZ_{k,t}\ ,\ \sum_{m,l=1}^n(\sigma_{im,t}dW_{ml,t}Q_{lj}+\sigma_{jm,t}dW_{ml,t}Q_{li})> \\ &=<\sum_{k=1}^n \sigma_{hk,t}(\sqrt{1-\rho'\rho}dB_{k,t}+\sum_{p=1}^n dW_{kp,t}\rho_p)\ ,\ \sum_{m,l=1}^n(\sigma_{im,t}dW_{ml,t}Q_{lj}+ \end{aligned}$$

$$\sigma_{jm,t} dW_{ml,t} Q_{li}) > . \tag{4.7}$$

Since each component of the vector B_t is independent with the one matrix $\tilde{W}_t$, then $< dB_{k,t}\ ,\ dW_{sm,t} >= 0\ \forall\ k,\ s$ and m.
So, using

$$< dW_{kp,t}\ ,\ dW_{ml,t} >= \begin{cases} 0 \text{ si } (k\ ,\ p) \neq (m\ ,\ l) \\ dt \text{ otherwise} \end{cases}, \tag{4.8}$$

we have

$$\begin{aligned}
(4.7) &= < \sum_{k=1}^{n} \sigma_{hk,t} \sum_{p=1}^{n} dW_{kp,t}\rho_p\ ,\ \sum_{m,l=1}^{n} (\sigma_{im,t} dW_{ml,t} Q_{lj} + \sigma_{jm,t} dW_{ml,t} Q_{li}) > \\
&= \sum_{k,l=1}^{n} (\sigma_{hk,t}\sigma_{ik,t} Q_{lj}\rho_l dt) + \sum_{k,l=1}^{n} (\sigma_{hk,t}\sigma_{jk,t} Q_{li}\rho_l dt) \\
&= \sum_{k,l=1}^{n} (\sigma_{hk,t}\sigma_{ik,t} Q_{lj}\rho_l dt) + \sum_{k,l=1}^{n} (\sigma_{hk,t}\sigma_{jk,t} Q_{li}\rho_l dt) \\
&= \left(\Gamma_{hi,t} \sum_{l=1}^{n} Q_{lj}\rho_l + \Gamma_{hj,t} \sum_{l=1}^{n} Q_{li}\rho_l \right) dt, \text{ through (4.5).}
\end{aligned}$$

□

We assume that the model also checks:

i) the correlation between each component of the vector $\log S_t$ and the one matrix Γ_t is negative (volatility leverage effect),
ii) the correlation between each yield of the log basket $\log S_{p,t}$ and the one correlations $\zeta_{pq,t}$, $p, q = 1, ..., n$ and $p \neq q$ is negative (correlation leverage effect) where $\zeta_{pq,t}$ is the correlation between $\Gamma_{pp,t}$ and $\Gamma_{qq,t}$ defined by

$$\zeta_{pq,t} = \frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}. \tag{4.9}$$

Theorem 4.7. *The expressions of correlations between each component of the vector* $\log S_t$ *and the one matrix* Γ_t *at time* t *are defined by:*

$$corr((\log S_{i,t})^c\ ,\ (\Gamma_{ii,t})^c) = \frac{\sum_{l=1}^{n} Q_{li}\rho_l}{\sqrt{\sum_{l=1}^{n} Q_{li}^2}}, i = 1, ..., n. \tag{4.10}$$

So, the sign and magnitude of the skew effect are determined by both the matrix Q *and the vector* ρ.

Proof. Let be $i \in [|\ 1, n\ |]$. The standard deviation of $(\log S_{i,t})^c$ is $\sqrt{\Gamma_{ii,t}}$. Indeed, we have

$d(\log S_t)^c = \left(\mu + \begin{bmatrix} tr(D_1\Gamma_t) \\ : \\ tr(D_n\Gamma_t) \end{bmatrix} \right) dt + \sqrt{\Gamma_t} dZ_t$. So $Var(d(\log S_t)^c) = \Gamma_t dt$.

The standard deviation of $(\Gamma_{ii,t})^c$ is

$$\sqrt{< (\Gamma_{ii,t})^c >} = 2\sqrt{\Gamma_{ii,t}\sum_{l=1}^{n} Q_{li}^2}. \tag{4.11}$$

Indeed, we have

$$\begin{aligned}
Var((d\Gamma_{ii,t})^c) &= < d(\Gamma_{ii,t})^c \ , \ d(\Gamma_{ii,t})^c > \\
&= < \sum_{m,l=1}^{n} (\sigma_{im,t} dW_{ml,t} Q_{li} + \sigma_{im,t} dW_{ml,t} Q_{li}) \ , \\
&\quad \sum_{m,l=1}^{n} (\sigma_{im,t} dW_{ml,t} Q_{li} + \sigma_{im,t} dW_{ml,t} Q_{li}) > \\
&= \sum_{m,l,p,q=1}^{n} (4\sigma_{im,t}\sigma_{ip,t} < dW_{ml,t} \ , \ dW_{pq,t} > Q_{li} Q_{qi}) \\
&= \sum_{m,l=i}^{n} (4\sigma_{im,t}\sigma_{im,t} 1 dt Q_{li}^2), \text{ through (4.8)} \\
&= 4\Gamma_{ii,t}\sum_{l=1}^{n} Q_{li}^2 dt, \text{ through (4.5).}
\end{aligned}$$

By using the correlation formula on an affine line $r_{x,y} = \frac{cov(x \ , \ y)}{\sigma_x \sigma_y}$, we have $corr((\log S_{i,t})^c \ , \ (\Gamma_{ii,t})^c) = \frac{<(\log S_{i,t})^c \ , \ (\Gamma_{ii,t})^c)>}{\sqrt{\Gamma_{ii,t}}\sqrt{<(\Gamma_{ii,t})^c>}}$.
From the equations (4.4) and (4.11) and imposing $i, j, h = 1$, we have

$$\begin{aligned}
corr((\log S_{i,t})^c \ , \ (\Gamma_{ii,t})^c) &= \frac{2\Gamma_{ii,t}\sum_{l=1}^{n} Q_{li}\rho_l}{\sqrt{\Gamma_{ii,t}}(2\sqrt{\Gamma_{ii,t}\sum_{l=1}^{n} Q_{li}^2})} \\
&= \frac{\sum_{l=1}^{n} Q_{li}\rho_l}{\sum_{l=1}^{n} Q_{li}^2}.
\end{aligned}$$

□

4.3. **Dependence between yield and its correlations.**

Theorem 4.8. *The expressions of covariances between each yield noise of the basket* $\log S_{p,t}$ *and the correlations noises* $\zeta_{pq,t}$, $p, q = 1, ..., n$ *and* $p \neq q$ *are given by:*

$$cov(d(\log S_{p,t})^c \ , \ d(\zeta_{pq,t})^c) = \left(\sum_{l=1}^{n} Q_{lq}\rho_l\right)\sqrt{\frac{\Gamma_{pp,t}}{\Gamma_{qq,t}}}(1-\zeta_{pq,t}^2)dt, \tag{4.12}$$

with $(\zeta_{pq,t})^c$ *is the continuous part of* $\zeta_{pq,t}$.

Proof. Let be $p, q \in \{1, ..., n\}$, $p \neq q$.
Applying Ito's formula on Levy's process $f(\Gamma_t) = \frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}$, we get

$$\begin{aligned} d\zeta_{pq,t} &= \frac{d\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}} - \frac{\frac{1}{2}\frac{1}{\sqrt{\Gamma_{pp,t}}}}{\Gamma_{pp,t}}\left(\frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{qq,t}}}\right) d\Gamma^c_{pp,t} - \frac{\frac{1}{2}\frac{1}{\sqrt{\Gamma_{qq,t}}}}{\Gamma_{qq,t}}\left(\frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}}}\right) d\Gamma^c_{qq,t} \\ &\quad + \frac{1}{2}\sum_{i,j,k,l=1}^{n} \frac{\partial^2 f(\Gamma_t)}{\partial\Gamma_{ij,t}\partial\Gamma_{kl,t}} d < (\Gamma_{ij})^c \ , \ (\Gamma_{kl})^c >_t + [f(\Gamma_{t^-} + \sqrt{\Gamma_{t^-}}J + \\ &\quad J'\sqrt{\Gamma_{t^-}} + JJ') - f(\Gamma_{t^-})]dN_t. \end{aligned}$$

So

$$\begin{aligned} d(\zeta_{pq,t})^c &= \frac{d\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}} - \frac{1}{2\sqrt{\Gamma_{pp,t}}\Gamma_{pp,t}}\left(\frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{qq,t}}}\right) d\Gamma^c_{pp,t} - \frac{1}{2\sqrt{\Gamma_{qq,t}}\Gamma_{qq,t}} \\ &\quad \left(\frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}}}\right) d\Gamma^c_{qq,t} + \frac{1}{2}\sum_{i,j,k,l=1}^{n} \frac{\partial^2 f(\Gamma_t)}{\partial\Gamma_{ij,t}\partial\Gamma_{kl,t}} d < (\Gamma_{ij})^c \ , \ (\Gamma_{kl})^c >_t . \end{aligned} \tag{4.13}$$

However

$$\begin{aligned} & d < (\Gamma_{ij})^c \ , \ (\Gamma_{kl})^c >_t \\ & =< d(\Gamma_{ij,t})^c \ , \ d(\Gamma_{kl,t})^c > \\ & =< \sum_{m,r=1}^{n} (\sigma_{im,t}dW_{mr,t}Q_{rj} + \sigma_{jm,t}dW_{mr,t}Q_{ri}) \ , \ \sum_{m,r=1}^{n} (\sigma_{km,t}dW_{mr,t}Q_{rl} + \\ & \sigma_{lm,t}dW_{mr,t}Q_{rk}) > \\ & = \sum_{m,r,p,s=1}^{n} \sigma_{im}\sigma_{kp,t}dW_{mr,t}dW_{ps,t}Q_{rj}Q_{sl} + \sigma_{im,t}\sigma_{lp,t}dW_{mr,t}dW_{ps,t}Q_{rj}Q_{sk} + \\ & \sigma_{jm,t}\sigma_{kp,t}dW_{mr,t}dW_{ps,t}Q_{ri}Q_{sl} + \sigma_{jm,t}\sigma_{lp,t}dW_{mr,t}dW_{ps,t}Q_{ri}Q_{sk} \\ & = \sum_{m,r=1}^{n} \sigma_{im,t}\sigma_{km,t}dtQ_{rj}Q_{rl} + \sigma_{im,t}\sigma_{lm,t}dtQ_{rj}Q_{rk} + \sigma_{jm,t}\sigma_{km,t}dtQ_{ri}Q_{rl} + \\ & \sigma_{jm,t}\sigma_{lm,t}dtQ_{ri}Q_{rk}, \text{ through (4.4)} \\ & = \Gamma_{ik,t}dt\sum_{r=1}^{n} Q_{rj}Q_{rl} + \Gamma_{il,t}dt\sum_{r=1}^{n} Q_{rj}Q_{rk} + \Gamma_{jk,t}dt\sum_{r=1}^{n} Q_{ri}Q_{rl} + \\ & \Gamma_{jl,t}dt\sum_{r=1}^{n} Q_{ri}Q_{rk}, \text{ through (4.5).} \end{aligned}$$

Then (4.13) is equal to

$$\frac{d(\Gamma_{pq,t})^c}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}} - \frac{1}{2}\frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}\left(\frac{d(\Gamma_{pp,t})^c}{\Gamma_{pp,t}} + \frac{d(\Gamma_{qq,t})^c}{\Gamma_{qq,t}}\right) + \frac{1}{2}\sum_{i,j,k,l=1}^{n} \frac{\partial^2 f(\Gamma_t)}{\partial\Gamma_{ij,t}\partial\Gamma_{kl,t}}$$

$$\left[\Gamma_{ik,t}\sum_{r=1}^{n}Q_{rj}Q_{rl}+\Gamma_{il,t}\sum_{r=1}^{n}Q_{rj}Q_{rk}+\Gamma_{jk,t}\sum_{r=1}^{n}Q_{ri}Q_{rl}+\Gamma_{jl,t}\sum_{r=1}^{n}Q_{ri}Q_{rk}\right]dt. \tag{4.14}$$

Hence

$$\begin{aligned}
&cov(d(\log S_{i,t})^c\ ,\ d(\zeta_{pq,t})^c)\\
&=\frac{cov(d(\log S_{i,t})^c\ ,\ d(\Gamma_{pq,t})^c)}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}-\frac{1}{2}\frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}\\
&\left(\frac{cov(d(\log S_{i,t})^c\ ,\ d(\Gamma_{pp,t})^c)}{\Gamma_{pp,t}}+\frac{cov(d(\log S_{i,t})^c\ ,\ d(\Gamma_{qq,t})^c)}{\Gamma_{qq,t}}\right)\\
&=\frac{\Gamma_{ip,t}\sum\limits_{l=1}^{n}Q_{lq}\rho_l+\Gamma_{iq,t}\sum\limits_{l=1}^{n}Q_{lp}\rho_l}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}dt-\frac{1}{2}\frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}\\
&\left(\frac{2\Gamma_{ip,t}\sum\limits_{l=1}^{n}Q_{lp}\rho_l}{\Gamma_{pp,t}}dt+\frac{2\Gamma_{iq,t}\sum\limits_{l=1}^{n}Q_{lq}\rho_l}{\Gamma_{qq,t}}dt\right)\text{, through (4.4)}
\end{aligned}$$

$$=\frac{(\Gamma_{qq,t}\Gamma_{ip,t}-\Gamma_{pq,t}\Gamma_{iq,t})\sum\limits_{l=1}^{n}Q_{lq}\rho_l dt}{\Gamma_{qq,t}\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}+\frac{(\Gamma_{pp,t}\Gamma_{iq,t}-\Gamma_{pq,t}\Gamma_{ip,t})\sum\limits_{l=1}^{n}Q_{lp}\rho_l dt}{\Gamma_{pp,t}\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}. \tag{4.15}$$

Assuming $i=p$, we have

$$\begin{aligned}
(4.15)\quad &= \frac{\Gamma_{qq,t}\Gamma_{pp,t}\sum\limits_{l=1}^{n}Q_{lq}\rho_l-\Gamma_{pq,t}\Gamma_{pq,t}\sum\limits_{l=1}^{n}Q_{lq}\rho_l}{\Gamma_{qq,t}\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}dt\\
&= \sum_{l=1}^{n}Q_{lq}\rho_l\left(\sqrt{\frac{\Gamma_{pp,t}}{\Gamma_{qq,t}}}-\frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}\frac{\Gamma_{pq,t}}{\Gamma_{qq,t}}\right)dt\\
&= \sum_{l=1}^{n}Q_{lq}\rho_l\sqrt{\frac{\Gamma_{pp,t}}{\Gamma_{qq,t}}}(1-\zeta_{pq,t}^2)dt\text{, through (4.9).}
\end{aligned}$$

□

4.4. **Up-jump and down-jump.** On the one hand, our jump process ψ_t can make the volatility Γ_t to down jump even it respects its positivity through the expression of Γ_t of the form (1.2). On the other hand, a jump on the volatility will cause a jump on the yield with the direction and frequency which depend the parameters φ_i where the φ_i are the components of the vector φ.

4.5. **Return to average.** It is a very negative bias towards the average. The deterministic part of O.U process on $\mathbb{R}$ $\kappa(\theta - x_t)$ defines its return. The form is imitated by the recall force of a spring expressed by $\parallel \vec{F} \parallel = k|L-L_0|$ where k is the stiffness coefficient of spring and L_0 is its empty length.

4.6. **Clusters of volatility.** The level of volatility depends essentially the level of volatility at the previous moment, which makes it possible to model periods of high volatility and periods of low volatility. Hence, we observe a high volatility no punctually but on the moment interval or the periods of high volatility generally followed by periods of high volatility. Its phenomena calls the clusters of volatility. The term $\sqrt{\Gamma_t}$ on the diffusion in the dynamic of volatility can to render account the periods of high volatility.

5. Application

We resume at first the C-GMM method (dependent data) to estimate the model parameters but the details of its method are in the references [9, 10, 14]. And we present after the results of estimations of the CAC40 and SP500 indexes by using the model with its stylized facts.

5.1. **Use of C-GMM method.** Let h_t be the continuum of moment conditions defined by

$$h_t = e^{\varsigma<w,Y_{t+1}-Y_t>} - X \tag{5.1}$$

with $w \in \mathbb{R}^n$, X is a stochastic function of the process parameters and $g(Y_t)$ be an arbitrary instrument. We determine X by the relation :

$$\mathbb{E}(h_t g(Y_t)) = 0 \Leftrightarrow \mathbb{E}\left(e^{\varsigma<w,Y_{t+1}-Y_t>}/Y_t\right) - \mathbb{E}(Xg(Y_t)) = 0. \tag{5.2}$$

Chacko and Viceira (1999) showed that

$$X = \mathbb{E}\left(e^{\varsigma<w,Y_{t+1}-Y_t>}/Y_t\right). \tag{5.3}$$

In our case,

$$X = e^{C(1)}\mathbb{E}\left(e^{<A(1),\Gamma_t>}/Y_t\right) = e^{C(1)}\Psi_{\Gamma_0}(-\varsigma A(1),t) \tag{5.4}$$

with $A(1)$ and $C(1)$) are the deterministic functions of the characteristic function $\Psi_{\log S_t}(w,\tau) = e^{tr(A(\tau)\Gamma_t)+B(\tau)Y_t+C(\tau)}$.
(5.2) is well defined if $g(Y_t) = 1$ or (Y_t) and (Γ_t) are independent processes. But the second Assumption is not valid for our model. So, we suppose that $g(Y_t) = 1$.
Let now, $\hat{h}(.)$ be the empirical moment of h from $\mathbb{R}^n$ to $\mathbb{C}$ defined by

$$\hat{h}(w,\theta) = \frac{1}{T}\sum_{t=1}^{T} h_t(w,\theta) \tag{5.5}$$

where θ is the parameter vector of the model.
The C-GMM estimator of θ is defined by

$$\hat{\theta} = \arg\underset{\theta}{min} \parallel K^{-\frac{1}{2}}\hat{h}_T(\theta) \parallel \tag{5.6}$$

where K is the covariance operator and $\parallel . \parallel$ is the norm defined by

$$\parallel f \parallel^2 = \int_{\mathbb{R}^n} f(\lambda)\overline{f(\lambda)}\pi(\lambda)d\lambda \tag{5.7}$$

with π is a probability measure.
Carrasco (2007) have shown that the operator K can write as

$$Kf(w) = \int k(w,\lambda)f(\lambda)\pi(\lambda)d\lambda \tag{5.8}$$

with w, $\lambda \in \mathbb{R}^n$, k is the coefficient defined by

$$k(w,\lambda) = \sum_{j=-\infty}^{+\infty} \mathbb{E}^{\theta_0}\left(h_t(w,\theta_0)\overline{h_{t-j}(\lambda,\theta_0)}\right) \tag{5.9}$$

where θ_0 is the true value of θ.
To build K, Carrasco (2007) proposed the following steps. The first step is to find

$$\hat{\theta}_1 = \arg\min_{\theta} \parallel \hat{h}_T(\theta) \parallel . \tag{5.10}$$

The second step consists to estimate the coefficient k by

$$k(w_s,w_r,w_v,w_w) = \frac{T}{T-q}\sum_{j=-T+1}^{T-1}\omega\left(\frac{j}{S_T}\right)\hat{\Gamma}_T(j) \tag{5.11}$$

with
$$\hat{\Gamma}_T(j) = \begin{cases} \frac{1}{T}\sum\limits_{t=j+1}^{T} h_t(w_s,w_r,\hat{\theta}_1)\overline{h_{t-j}(w_v,w_w,\hat{\theta}_1)},\ j\geq 0 \\ \frac{1}{T}\sum\limits_{t=-j+1}^{T} h_{t+j}(w_s,w_r,\hat{\theta}_1)\overline{h_t(w_v,w_w,\hat{\theta}_1)},\ j<0 \end{cases}$$
where $\omega(.)$ is the coefficient satisfying the conditions defined in the work of Carrasco (2007) and S_T is the bandwidth parameter of ω.
When K is estimated, the minimization of (5.6) requires the inverse of K. Carrasco (2007) used Tikhonov's approximation which generalizes the inverse of K. Let α be a strictly positive parameter, then K^{-1} is replaced by $(K^{\alpha})^{-1} = (K^2+\alpha I)^{-1}K$. So the optimal C-GMM estimator of θ is obtained by

$$\hat{\theta} = \arg\min_{\theta} \parallel (K^{\alpha})^{-1}\hat{h}_T(\theta) \parallel . \tag{5.12}$$

<u>Asymptotic convergence:</u> $\sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{\mathcal{L}} N\left(0, \left(< \mathbb{E}^{\theta_0}(\nabla_\theta h), E^{\theta_0}(\nabla_\theta h) >_K\right)^{-1}\right)$ when T and $T^a(\alpha_T)^{\frac{5}{4}}$ tends to infinity and α_T tends to 0 with $\nabla_\theta h$ is the Jacobian matrix of h(.).
Let us

$$Uh_t(w,\hat{\theta}_T^1) = w(0)\overline{h_t(w,\hat{\theta}_T^1)} + \sum_{j=1}^{T} w\left(\frac{j}{S_T}\right)\left(\overline{h_{t-j}(w,\hat{\theta}_T^1)} + \overline{h_{t+j}(w,\hat{\theta}_T^1)}\right) \tag{5.13}$$

with the convention $h_t(w,\hat{\theta}_T^1) = 0$ if $t \leq 0$ or $t > T$ and in the case where (h_t) is uncorrelated, the form simplifies $Uh_t = \overline{h_t}$.
Carrasco (2007) have shown that the resolution of (5.12) is equivalent to

$$\min_{\theta} \underline{W'}(\theta)(\alpha_T I_T + C^2)^{-1}\underline{V}(\theta) \tag{5.14}$$

with C is the $T\times T$ dimensional matrix whose the components are $\frac{c_{tl}}{T-q}$ where q is the number of parameters of θ; I_T is the $T\times T$ dimensional identity matrix; $\underline{V(\theta)} = (V_1(\theta),...,V_T(\theta))'$ and $\underline{W(\theta)} = (W_1(\theta),...,W_T(\theta))'$ are the T-dimensional vectors where

$$V_t(\theta) = \int U h_t(w,\hat\theta_T^1)\hat h_T(w,\theta)\pi(w)dw; \quad (5.15)$$

$$W_t(\theta) = \int h_t(w,\hat\theta_T^1)\overline{\hat h_T(w,\theta)}\pi(w)dw; \quad (5.16)$$

$$c_{tl} = \int U h_t(w,\hat\theta_T^1)h_l(w,\hat\theta_T^1)\pi(w)dw. \quad (5.17)$$

And the C-GMM estimator $< \mathbb{E}^{\theta_0}(\nabla_\theta h), E^{\theta_0}(\nabla_\theta h) >_K$ is given by:

$$< \nabla_\theta \hat h_T(\hat\theta_T), (K^{\alpha_T})^{-1}\nabla_\theta \hat h_T(\hat\theta_T) > = \frac{1}{T-q}\underline{W'}(\hat\theta_T)(\alpha_T I_T + C^2)^{-1}\underline{V}(\hat\theta_T) \quad (5.18)$$

with C is the matrix defined above, $\underline{V} = (V_1, ..., V_T)'$ and $\underline{W} = (W_1, ..., W_T)'$ are the $T \times q$ dimensional matrices where

$$(V_t)_j = \int U h_t(w,\hat\theta_T^1)\nabla_{\theta_j}\hat h_T(w,\hat\theta_T)\pi(w)dw; \quad (5.19)$$

$$(W_t)_j = \int h_t(w,\hat\theta_T)\overline{\nabla_{\theta_j}\hat h_T(w,\hat\theta_T)}\pi(w)dw. \quad (5.20)$$

5.2. **Gradient of the characteristic function.** Let $Y_t = log(S_t)$ be the yield of underlying and β be a component of the vector θ. Denote by $\partial_\beta f(\theta)$ the partial derivative of the function $f(\theta)$. We have

$$\partial_\beta \Psi_{Y_t,\Gamma_t}(w,\tau) = (tr(\partial_\beta A(\tau)\Gamma_t) + \partial_\beta C(\tau))\Psi_{Y_t,\Gamma_t}(w,\tau) \quad (5.21)$$

with

$$\partial_\beta A(\tau) = -A_{22}(\tau)^{-1}\partial_\beta A_{22}(\tau)A(\tau) + A_{22}(\tau)^{-1}\partial_\beta A_{21}(\tau) \quad (5.22)$$

$$\begin{aligned}\partial_\beta C(\tau) &= tr\left((\partial_\beta r)\tau\check{1}(\varsigma\gamma)' - \frac{\nu}{2}\partial_\beta log(A_{22}(\tau))\right) - \frac{\nu\tau}{2} \\ & tr\left(\frac{\partial_\beta\Phi + \partial_\beta\Phi'}{2} + \frac{\varsigma w\partial_\beta(\rho'\sqrt{Q'Q}) + \partial_\beta(\sqrt{Q'Q}\rho)\varsigma w'}{2}\right) \\ & +\partial_\beta\left[\lambda\tau\left(e^{tr\left[\omega\mu^{-1}\left(\frac{1}{2}(m\tilde{1})^2 + \sqrt{\Gamma_t}(m\tilde{1}) + \frac{1}{2}\sigma^2\Gamma_t\omega\right) - \frac{n}{2}\log\mu\right]} - 1\right)\right]. \quad (5.23)\end{aligned}$$

Let us $G = \begin{bmatrix} \frac{(\Phi+\varsigma w\rho'\sqrt{Q'Q})+(\Phi+\varsigma w\rho'\sqrt{Q'Q})'}{2} & -2Q'Q \\ -\frac{1}{2}\sum_{i=1}^n \varsigma w_i e_{ii} + \frac{1}{2}(\varsigma w)(\varsigma w)' & -\frac{(\Phi+\varsigma w\rho'\sqrt{Q'Q})+(\Phi+\varsigma w\rho'\sqrt{Q'Q})'}{2}\end{bmatrix}$.

If $\beta = \Phi_{kl}$, then we have

$$\partial_\beta G = \begin{bmatrix} \frac{e_{kl}+e'_{kl}}{2} & 0 \\ 0 & -\frac{e_{kl}+e'_{kl}}{2}\end{bmatrix}. \quad (5.24)$$

If $\beta = Q_{kl}$ where $\sqrt{Q'Q} = (Q_{kl})_{kl}$, then we have

$$\partial_\beta G = \begin{bmatrix} \frac{e'_{kl}\rho\varsigma w' + \varsigma w\rho' e_{kl}}{2} & -2(\sqrt{Q'Q}e_{kl} + e'_{kl}\sqrt{Q'Q}) \\ 0 & -\frac{e'_{kl}\rho\varsigma w' + \varsigma w\rho' e_{kl}}{2}\end{bmatrix}. \quad (5.25)$$

If $\beta = \rho_l$, then we have

$$\partial_\beta G = \begin{bmatrix} \frac{\sqrt{Q'Q}e_l\varsigma w' + \varsigma w e'_l\sqrt{Q'Q}}{2} & 0 \\ 0 & \frac{\sqrt{Q'Q}e_l\varsigma w' + \varsigma w e'_l\sqrt{Q'Q}}{2}\end{bmatrix} \quad (5.26)$$

where (e_l) is the canonical basis of $\mathbb{R}^n$ and (e_{kl}) is the canonical basis of $\mathcal{M}_n(\mathbb{R})$. In addition

$$\partial_\beta e^{\tau G} = D_{exp_{\tau G}}\partial_\beta(\tau G) \tag{5.27}$$

where $D_{expX} = e^X \frac{I-e^{-ad_X}}{ad_X}$ and $ad_X = [X,Y] = XY - YX$.

Let us $P_{22}L = L_{22}$ with $L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$.

We have

$$\partial_\beta log(A_{22}(\tau)) = D_{log,A_{22}(\tau)}(P_{22}D_{exp_{\tau G}}\partial_\beta(\tau G)) \tag{5.28}$$

where $D_{f,X}(H) = PM_{log} \circ (P^{-1}HP)P^{-1}$, P is the matrix associated of the eigenvector of X, $A \circ B = (a_{ij}b_{ij})$ where $A = (a_{ij})$ and $B = (b_{ij})$ are the matrix in $\mathcal{M}_n(\mathbb{R})$ and $M_{log} = \begin{bmatrix} \frac{1}{\lambda_1} & \frac{log\lambda_1 - log\lambda_2}{\lambda_1 - \lambda_2} \\ \frac{log\lambda_1 - log\lambda_2}{\lambda_1 - \lambda_2} & \frac{1}{\lambda_2} \end{bmatrix}$ for $n = 2$.

We have also

$$e^{C(\tau)}\Psi_{\Gamma_0}(-\varsigma A(\tau), t) = e^{tr(B(t)\Gamma_0) + c(t) + C(\tau)} \tag{5.29}$$

with

$$\begin{aligned}
B(t) &= (A(\tau)B_{12}(t) + B_{22}(t))^{-1}(A(\tau)B_{11}(t) + B_{21}(t)); \qquad (5.30)\\
c(t) &= -\frac{\nu}{2}tr\left[log(A(\tau)B_{12}(t) + B_{22}(t)) + t\frac{\Phi + \Phi'}{2}\right] - \lambda t \\
&\quad + \lambda \int_0^t e^{tr\left[B(u)\Delta(u)^{-1}\left((m\tilde{1})^2 + 2\sqrt{\Gamma_{t^-}}(m\tilde{1}) + 2\Gamma_{t^-}B(u)\sigma^2\right) - \frac{n}{2}\log\Delta(u)\right]}du \qquad (5.31)\\
C(\tau) &= tr\left[r\check{1}\tau(\varsigma\gamma)' - \frac{\nu}{2}\left(\log A_{22}(\tau) + \tau\frac{\Phi + \Phi'}{2} + \tau\frac{\varsigma w\rho'\sqrt{Q'Q} + \sqrt{Q'Q}\rho\varsigma w'}{2}\right)\right] \\
&\quad + \lambda\tau\left(e^{tr\left[\omega\mu^{-1}\left(\frac{1}{2}(m\tilde{1})^2 + \sqrt{\Gamma_t}(m\tilde{1}) + \frac{1}{2}\sigma^2\Gamma_t\omega\right) - \frac{n}{2}\log\mu\right]} - 1\right). \qquad (5.32)
\end{aligned}$$

And

$$\partial_\beta e^{C(\tau)}\Psi_{\Gamma_0}(-\varsigma A(\tau), t) = e^{C(\tau)}\Psi_{\Gamma_0}(-\varsigma A(\tau), t)(tr(\partial_\beta B(t)\Gamma_0) + \partial_\beta c(t) + \partial_\beta C(\tau)) \tag{5.33}$$

with

$$\begin{aligned}
\partial_\beta B(t) &= -(A(\tau)B_{12}(t) + B_{22}(t))(\partial_\beta A(\tau)B_{12}(t) + A(\tau)\partial_\beta B_{12}(t) + \partial_\beta B_{22}(t))^{-1} \\
&\quad B(t) + (A(\tau)B_{12}(t) + B_{22}(t))^{-1}(\partial_\beta A(\tau)B_{11}(t) + B_{21}(t)); \qquad (5.34)\\
\partial_\beta c(t) &= -\frac{\nu}{2}tr[D_{log,A(\tau)B_{12}(t)+B_{22}(t)}(\partial_\beta A(\tau)B_{12}(t) + A(\tau)\partial_\beta B_{12}(t) + \partial_\beta B_{22}(t))] \\
&\quad + \partial_\beta \int_0^t \lambda e^{tr\left[B(u)\Delta(u)^{-1}\left((m\tilde{1})^2 + 2\sqrt{\Gamma_{t^-}}(m\tilde{1}) + 2\Gamma_{t^-}B(u)\sigma^2\right) - \frac{n}{2}\log\Delta(u)\right]} - \lambda du.
\end{aligned} \tag{5.35}$$

5.3. **Variation of correlation.** To see the correlation leverage effect on the graph, we need the expression of correlation noise. From the expressions of (4.6) and (4.14), we get

$$\begin{aligned}
d(\zeta_{12,t})^c &= (A_t\zeta_{12,t}^2 + B_t\zeta_{12,t} + C_t)dt \\
&\quad + \left(\frac{\sigma_{11,t}Q_{12} + \sigma_{21,t}Q_{11}}{\sqrt{\Gamma_{11}\Gamma_{22}}} - \left(\frac{\sigma_{11,t}Q_{11}}{\Gamma_{11}} + \frac{\sigma_{21,t}Q_{12}}{\Gamma_{22}}\right)\zeta_{12,t}\right)dW_{11,t}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\sigma_{11,t}Q_{22}+\sigma_{21,t}Q_{21}}{\sqrt{\Gamma_{11}\Gamma_{22}}}-\left(\frac{\sigma_{11,t}Q_{21}}{\Gamma_{11}}+\frac{\sigma_{21,t}Q_{22}}{\Gamma_{22}}\right)\zeta_{12,t}\right)dW_{12,t}\\
&+\left(\frac{\sigma_{12,t}Q_{12}+\sigma_{22,t}Q_{11}}{\sqrt{\Gamma_{11}\Gamma_{22}}}-\left(\frac{\sigma_{12,t}Q_{11}}{\Gamma_{11}}+\frac{\sigma_{22,t}Q_{12}}{\Gamma_{22}}\right)\zeta_{12,t}\right)dW_{21,t}\\
&+\left(\frac{\sigma_{12,t}Q_{22}+\sigma_{22,t}Q_{21}}{\sqrt{\Gamma_{11}\Gamma_{22}}}-\left(\frac{\sigma_{12,t}Q_{21}}{\Gamma_{11}}+\frac{\sigma_{22,t}Q_{22}}{\Gamma_{22}}\right)\zeta_{12,t}\right)dW_{22,t}
\end{aligned}
\tag{5.36}
$$

where

$$A_t = \frac{(Q_{11}Q_{21}+Q_{12}Q_{22})}{\sqrt{\Gamma_{11,t}\Gamma_{22,t}}}-\frac{\Phi_{12}\sqrt{\Gamma_{22,t}}}{\sqrt{\Gamma_{11,t}}}-\frac{\Phi_{21}\sqrt{\Gamma_{11,t}}}{\sqrt{\Gamma_{22,t}}} \tag{5.37}$$

$$B_t = -\frac{\nu(Q_{11}^2+Q_{12}^2)}{2\Gamma_{11,t}}-\frac{\nu(Q_{21}^2+Q_{22}^2)}{2\Gamma_{22,t}}+\frac{Q_{11}^2+Q_{21}^2}{2\Gamma_{11,t}}+\frac{Q_{12}^2+Q_{22}^2}{2\Gamma_{22,t}} \tag{5.38}$$

$$C_t = \frac{\nu(Q_{11}Q_{21}+Q_{12}Q_{22})}{\sqrt{\Gamma_{11,t}\Gamma_{22,t}}}-\frac{2(Q_{11}Q_{12}+Q_{21}Q_{22})}{\sqrt{\Gamma_{11,t}\Gamma_{22,t}}}+\frac{\Phi_{12}\sqrt{\Gamma_{22,t}}}{\sqrt{\Gamma_{11,t}}}+\frac{\Phi_{21}\sqrt{\Gamma_{11,t}}}{\sqrt{\Gamma_{22,t}}} \tag{5.39}$$

5.4. **Results of estimation.** We present successively in this part : the indexes with the data and the initial parameters used on the one hand; the technical of the simulation on the other hand; and finally, we given the results obtained.

5.4.1. *Monte Carlo study.* We used the daily CAC40 and SP500 indexes. For each stock, the time series start the January 03, 2017 and end the February 28, 2017 which are presented by the following figures 1 and 2.

We are restricted to two underlying ($n = 2$). The initial parameters used in the simulation are:

$\Gamma_0 = \begin{bmatrix} 0.0225 & -0.0054 \\ -0.0054 & 0.0144 \end{bmatrix}$, $\Phi = \begin{bmatrix} -5 & -0.5 \\ -0.5 & -5 \end{bmatrix}$; $\varphi = (-1,-1)$; $\rho = (-0.3,-0.4)$; $\nu = 15$; $m = 0.01$; $\sigma = 0.01$; $\lambda = 0.4$; $\alpha = 0.00225$; $r = 0.05$; $\sqrt{Q'Q} = \begin{bmatrix} 0.12015891 & -0.01131245 \\ -0.01131245 & 0.09515434 \end{bmatrix}$.

The matrix $\sqrt{Q'Q}$ is obtained by using the long-term relationship (4.3). It is a necessary condition for the process Γ_t to be stationary.

The table 1 shows the descriptives statistics of the data used.

The figures 3 and 4 displays the C-GMM method criterion.

The table 2 presents the C-GMM estimator $\hat{\theta}_1$ defined by the equation (5.10).

The results of the estimates of $\hat{\theta}$ with its standard deviations of errors are presented in the table 3.

The table 4 presents the two measures which evaluates the performance of estimation method.

The figures 5, 6, 7 and 8 give the characteristics and stylized facts captured by the model and show also the forecast of two courses CAC40 and SP500.

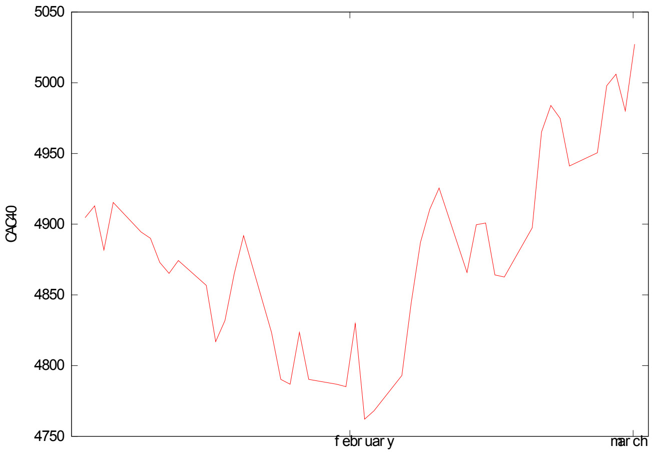

FIGURE 1. Historical Volume of the CAC40 Index

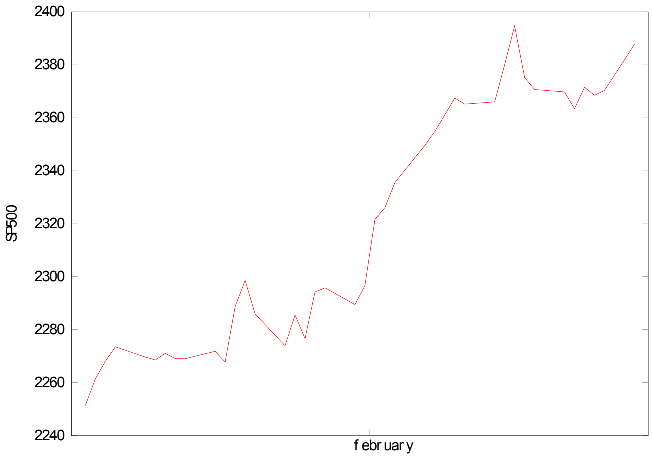

FIGURE 2. Historical Volume of the SP500 Index

5.4.2. *Empirical results.* We present the results studying the data by statistics descriptives analysis.

Table 1 : Analysis by descriptives statistics

Index	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
CAC40	4762	4844	4887	4885	4915	5027
SP500	2252	2272	2297	2317	2366	2395

The two underlying are no dispersed with compared to average. The price of CAC40 can be adjusted by the Gaussian distribution $N(4885, 66.76434)$) and the price of SP500 by $N(2317, 46.19202)$.

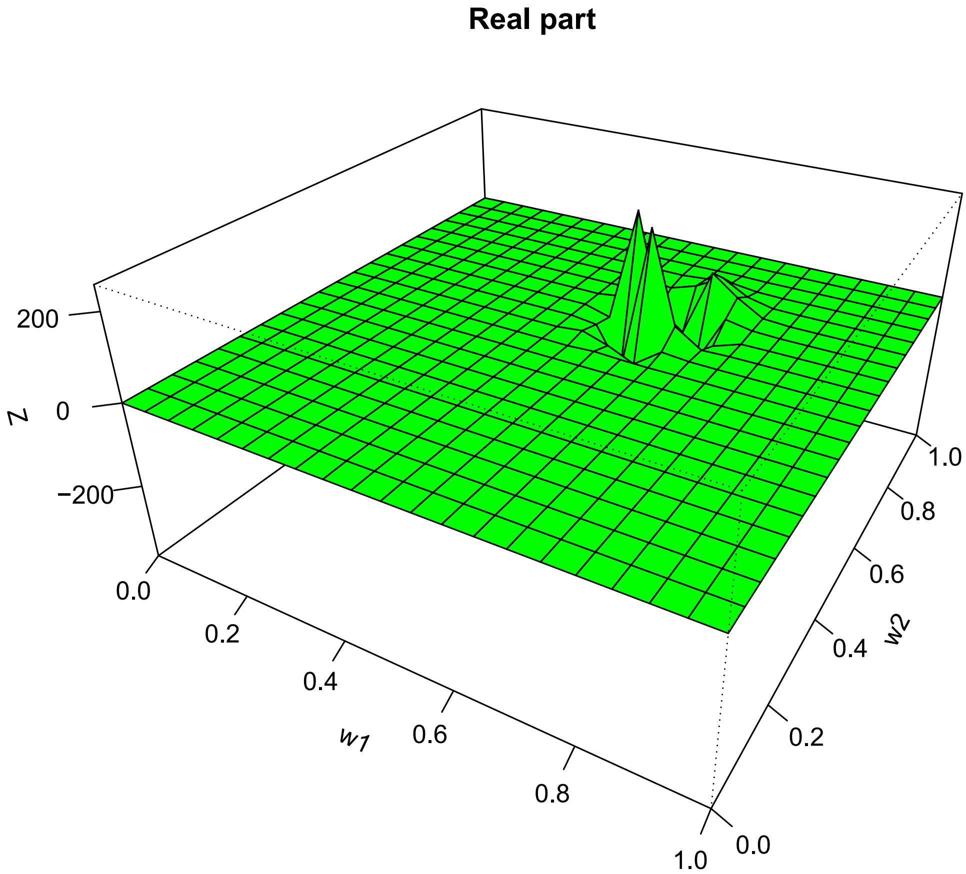

FIGURE 3. C-GMM estimation criterion

The figures 3 and 4 show us the values taken by : real and imaginary part of the empirical moment of continuum $\hat{h}$ defined in equation (5.5) and using the initial parameters presented in top. The figures show us that the minimizations of (5.10) and (5.6) exist.

Table 2 : C-GMM estimator $\hat{\theta}_1$

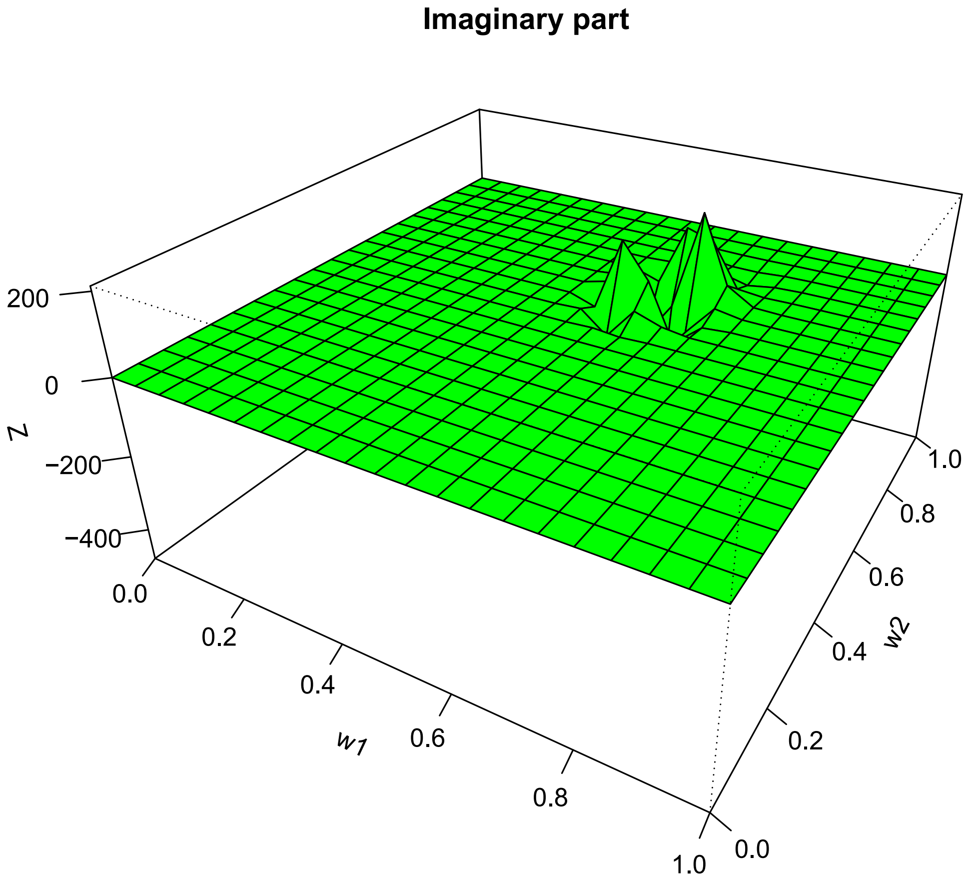

FIGURE 4. C-GMM estimation criterion

parameter	estimator
ρ_1	0,315492372
ρ_2	0,125872773
Q_{11}	0,106537319
$Q_{12} = Q_{21}$	0,000000000
Q_{22}	0,100000000
Φ_{11}	-5,042305318
Φ_{12}	-0,009478153
Φ_{21}	-0,010086226
Φ_{22}	-4,994793370
ν	15
r	0,046348362
λ	0,530854308
φ_1	-0,962462315
φ_2	-0,968014487
σ	0,240879750
m	-0,010000000

The minimum value for (5.10) is $\| \hat{h}_T(\hat{\theta}_1) \| = 3,965848.10^{-7}$.

Table 3 : C-GMM estimator $\hat{\theta}$

parameter	estimator	error standard deviation
ρ_1	-0,300319856	0,2945596
ρ_2	-0,400413190	0,2932194
Q_{11}	0,100000000	0,2606736
$Q_{12} = Q_{21}$	0,019316928	0,2624421
Q_{22}	0,100000000	0,2108252
Φ_{11}	-5,000272633	0,1096068
Φ_{12}	-0,052671480	0,1866816
Φ_{21}	-0,052652136	0,1865025
Φ_{22}	-5,000203409	0,1132604
ν	15	2,074395
r	0,010835953	1,383036
λ	0,400197918	2,072974
φ_1	-1,000097060	$4,903149.10^{-19}$
φ_2	-1,000086892	$4,903145.10^{-19}$
σ	0.013988008	$5,339199.10^{-90}$
m	0.009962586	$5,339199.10^{-90}$

The minimum value of (5.14) is $1,959312.10^{-9}$.

Table 4 : Mean Bias and RMSE (Root Mean Square Error)

parameter	Mean Bias	RMSE
ρ_1	0,00477225	0,2725201
ρ_2	-0,06413952	0,3444207
Q_{11}	0,03423131	0,1477892
$Q_{12} = Q_{21}$	0,1178646	0,2369885
Q_{22}	-0,05795468	0,2426288
Φ_{11}	0,02661671	0,1061759
Φ_{12}	-0,03949117	0,2323495
Φ_{21}	-0,05023514	0,1721152
Φ_{22}	0,001419603	0,1094503
ν	0,707043	1,921048
r	0,1616506	1,500114
λ	0.1063068	1,952471
φ_1	$-1,388808.10^{-19}$	$4,854741.10^{-19}$
φ_2	$2,516983.10^{-21}$	$6,363063.10^{-19}$
σ	$-2,287424.10^{-91}$	$5,595695.10^{-90}$
m	$2,424562.10^{-90}$	$6,135287.10^{-90}$

In modeling, we know that a model is pertinent if its volatility is very small, so we estimate σ very small. In the figures of volatilities of CAC40 and SP500 below (figure 6) where we simulate again and simultaneously the volatilities of the indexes CAC40 and SP500 with theirs yields and correlation, there exists a jump captured by the model in time 214. The impact of these jumps on the yields are presented in the figure 7. Here, the assets prices under the impact of jumps are difference compared with the one of WASC. At time 214, the asset values of CAC40

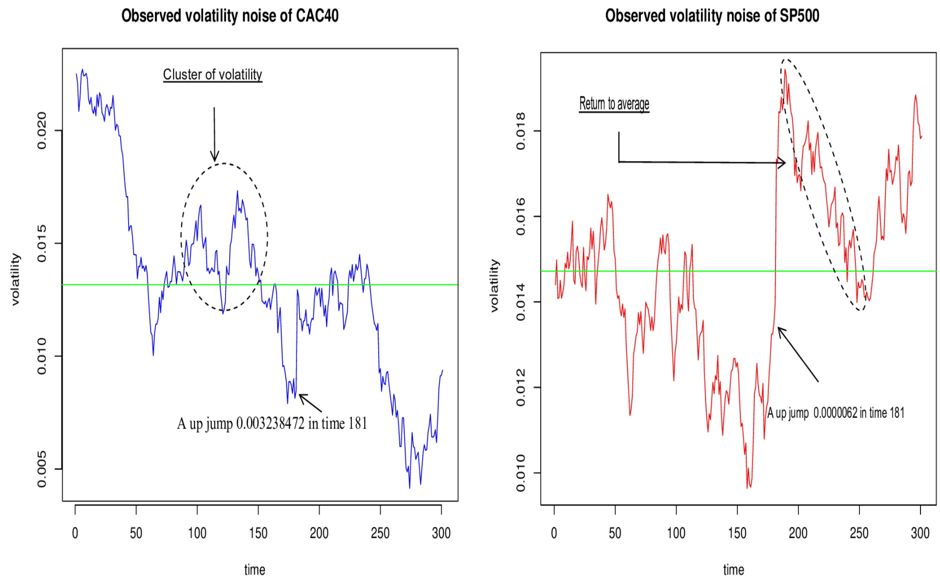

FIGURE 5. The characteristics captured by the model

and SP500 decrease respectively to -0,0062295 and -0,0086550 compared with the one of WASC.

Using the expressions of correlations defined by equation (4.10), we find $corr((\log S_{1,t})^c\ ,\ (\Gamma_{11,t})^c) = -0,3701 < 0$ and $corr((\log S_{2,t})^c\ ,\ (\Gamma_{22,t})^c) = -0,4496$ < 0 which we show us the asymmetrical correlation between the assets and its volatility. Graphically, the figure 6 associated with the figure 7 shows us this volatility leverage effect.

When we calculate numerically the sign of covariance defined in (4.12) $\sum_{l=1}^{n} Q_{lq}\rho_l$, we have find $-0,0458 < 0$ is the sign of covariance between the asset CAC40 with the correlation and $-0,0377 < 0$ is the one of SP500 with the correlation which we show us the asymmetrical correlation between the assets and its correlation. Graphically, the figure 7 associated with figure 8 shows us this correlation leverage effect.

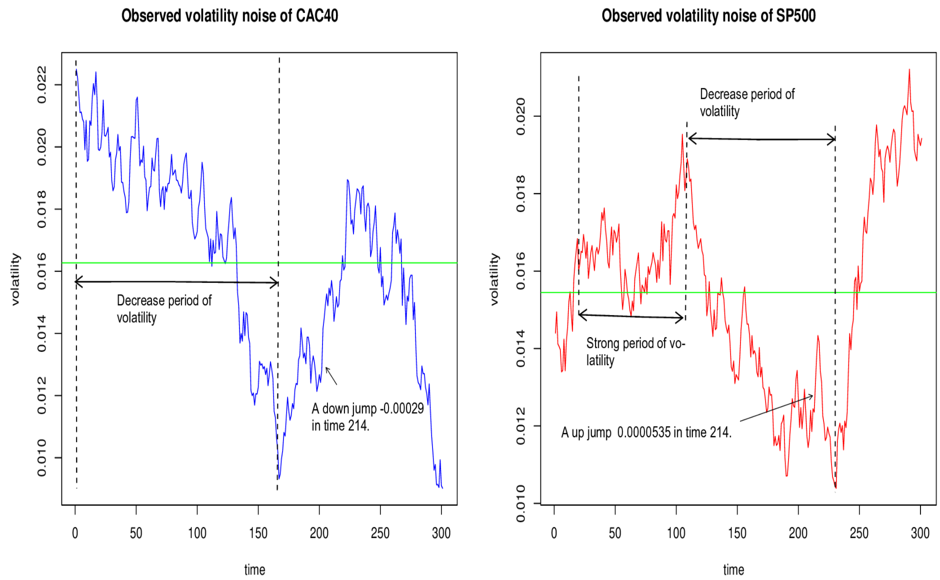

Figure 6. Volatilities noises of CAC40 and SP500

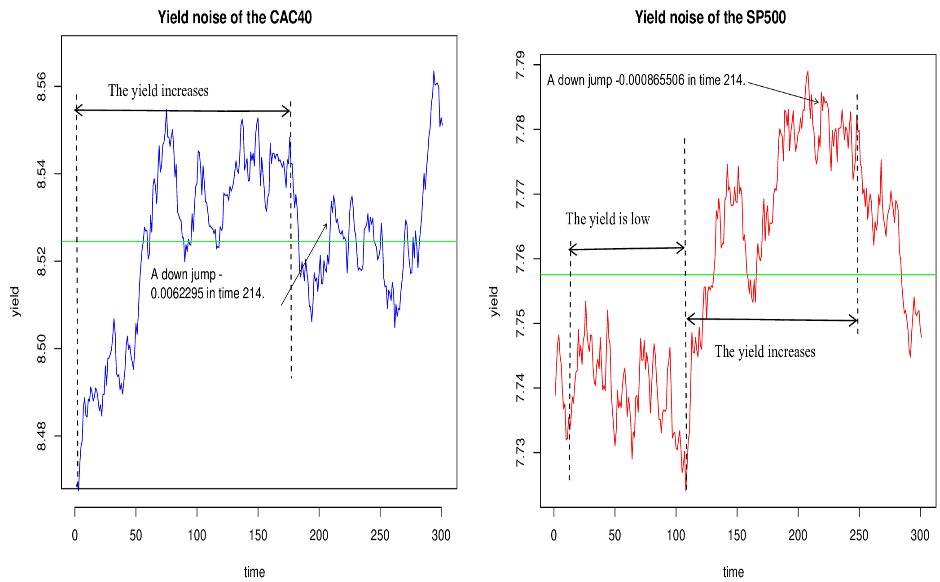

Figure 7. Yields noises of CAC40 and SP500

6. Discussion

We have developed a model which estimates the value of a basket carrying several underlying assets whose price is characterized primarily by jumps, clusters, return

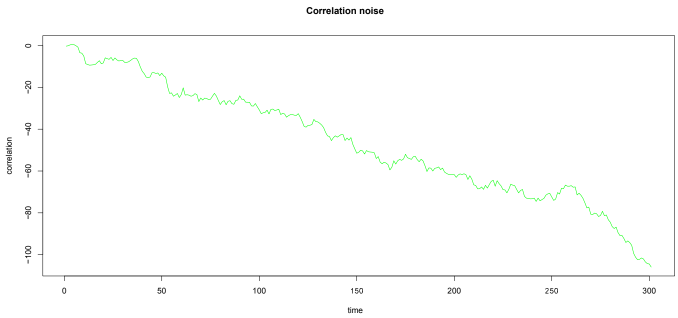

Figure 8. Correlation noise between the volatility of CAC40 with SP500

to average, volatility and correlation leverage effects. The implementation of the jumps process ψ_t in volatility stochastic and the new form respecting the stylized effects of the WASC specify the model. By taking σ estimated very small, the jumps exist and help the volatility of stochastic volatility of WASC to increase its value. In addition, if the impact frequencies of jumps φ_1 and φ_2 estimated are significant also, the model captures the value of assets perturbed. So, our model is always pertinent even if there exists the recent turbulences on the market because the jumps stabilize the value of σ estimated and capture the values perturbed by jumping. However, the WASC model can't to be the best model to estimate the assets prices perturbed because either it try to capture the values perturbed by obtaining σ estimated mostly big (there is an anomaly) or it don't capture the values perturbed.

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