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MEAN-VARIANCE PORTFOLIO SELECTION PROBLEM WITH TIME-DEPENDENT SALARY FOR DEFINED CONTRIBUTION PENSION SCHEME

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Abstract. This paper examines a mean-variance portfolio selection problem with time-dependent salary in the accumulation phase of a defined contribution (DC) pension scheme. It was assumed that the flow of contributions made by the pension plan member (PPM) are invested into a market that is characterized by a riskless and a risky assets. The aim of this paper are to find: the optimal portfolio, expected wealth of the PPM and efficient frontiers under the three utility functions (Quadratic Utility Function (QUF), Power Utility Function (PUF) and Exponential Utility Function (EUF)); the relationship between the three utility functions of PPM expected terminal wealth. The optimal portfolio processes and expected wealth for the PPM were established. The efficient frontier of a PPM portfolio in mean-standard deviation under QUF, PUF and EUF were established. Expected terminal wealth for the PPM at zero variance under QUF, EUF and PUF were obtained in this paper. It was found that a linear relationship exists between the utility functions of PPM expected terminal wealth (i.e., PUF and QUF, EUF and QUF and PUF and EUF).

1. Introduction

The Defined Contribution (DC) pension scheme was established by the Nigerian Pension Reform Act, 2004 which came into effect in June 25, 2004. The DC pension scheme has been in existence in several countries of the world. For example, in May 1981, Chile replaced its pension scheme known as Pay-As-You-Go (defined benefit) retirement scheme with a private managed system through making compulsory contribution into their retirement account. The Nigerian Pension Reform Act (NPRA) 2004 establishes a DC pension scheme for payment of retirement benefits of employees of the public service of the Federation, the Federal Capital Territory and the private sector (see Section 1(1) of the NPRA). Before the NPRA, the Nigerian pension scheme was poorly managed. This generated a lot of problems. The aims and objectives of the DC pension scheme are contained in Section 2 of the NPRA. These include (i) To ensure that every person who worked in either the public service of the Federation, Federal Capital Territory or private sector receives

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his retirement benefits as and when due; (ii) To assist improvident individuals by ensuring that they save in order to cater for their livelihood during old age; and (iii) To establish a uniform set of rules, regulations and standards for the administration and payments of retirement benefits for the public service of the Federation, Federal Capital Territory and the private sectors.

The DC pension scheme is contributory, fully funded, depend on individual accounts and level of risk. The Pension Fund Administrators (PFAs) have the statutory duty to manage the contributions of the PPMs under their care. The pension fund assets are held by Pension Fund Custodians (PFCs) under direct regulation process. The NPRA provides that both the employee and employer should make equal contributions into the DC pension scheme. Section 9(1) of the NPRA, provides that the employees should contribute a minimum of 7.5% of their Basic salary, Housing and Transport allowances and the employers should contribute 7.5% as well, of the employees salary, Housing and Transport allowances in case of both the public and the private sectors. An employer may elect to contribute on behalf of the employees provided that the total contributions should not be less than 15% of the Basic salary, Housing and Transport allowances of the employees (see Section 9(2) of the NPRA), for more on NPRA, see [28].

A mean-variance optimization is a quantitative method used to construct portfolios for the investors when the market is less volatile. The optimal investment allocation strategy can be found by solving a mean and variance optimization problem, see [24].

There are extensive literature that exist on the area of accumulation phase of DC pension plan and optimal investment strategies. See for instance, [6], [8], [16], [4], [2], [5], [10], [13], [25], [11], [9]. [21], [22], [23].

In the context of DC pension plans, the problem of finding the optimal investment strategy with time-dependent salary under mean-variance efficient approach has not been reported in published articles. [14] and [25] assumed a constant flow of contributions into the pension scheme which will not be applicable to a time-dependent salary earners in pension scheme. We assume that the contribution of the PPM grows as the salary grows over time. In the literature, the problem of determining the minimum variance on trading strategy in continuous-time framework has been studied by [20] via the Martingale approach. [1] used the same approach in a more general framework. [17] solved a mean-variance optimization problem in a discrete-time multi-period framework. [26] considered a mean-variance in a continuous-time framework. They show the possibility of transforming the difficult problem of mean-variance optimization problem into a tractable one, by embedding the original problem into a stochastic linear-quadratic control problem, that can be solved using standard methods. These approaches have been extended and used by many in the financial literature, see for instance, [25], [3], [14], [7], [15]. In this paper, we study a mean-variance approach to portfolio selection problem with time-dependent salary of a PPM in accumulation phase of a DC pension scheme under three utility functions. The utility functions were analyzed and the results compared to determined which them is more suitable for a scheme like pension plan. [24] considered a mean-variance portfolio selection problem with inflation hedging strategy for a defined contributory pension scheme. The efficient frontier was obtained for three asset classes which include cash account, stock and inflation-linked bond. It was found that inflation-linked bond is a suitable asset for hedging inflation risks in an investment portfolio. The paper assumed that the flow of contributions of the PPM is constant rate. In this paper, we assume that the salary of the PPM is a time-dependent process.

Highlights of this paper: (i) The portfolio values under mean-variance with time-dependent salary were obtained; (ii) The expected final wealth under EUF was found to coincides with the result obtained by [25] only when initial salary is a unit and growth rate null; (iii) The variance of expected final fund whether constant and deterministic salary under exponential utility function was found to be the same; (iv) The efficient frontier of the portfolios in mean-standard deviation was established under QUF, PUF and EUF; (v) The relationship between the expected final wealth for the PPM under PUF and QUF, EUF and QUF and PUF and EUF were established.

The remainder of this paper is organized as follows. In section 2, we present the problem formulation and financial market models. We also establish in this section, is the dynamics of the wealth process of PPM. In section 3, we present the mean-variance approach. In section 4, we present the optimization processes of our problem and expected wealth at time t and at the terminal period for the PPM. Also, in this section, we presents the efficient frontier of the mean-standard deviation under QUF. Section 5 present the optimization processes of our problem and expected wealth at time t and at the terminal period as well as the efficient frontier for the PPM under EUF. In section 6, we present the optimization processes of our problem and expected wealth at time t and at the terminal period for the PPM under PUF. Also, in this section, we presents the efficient frontier of the mean-standard deviation under PUF. In section 7, we present the special case of our models. Section 8 presents the numerical examples of our models. Finally, section 9 concludes the paper.

2. Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathbb{F}(\mathcal{F}) = \{\mathcal{F}_t : t \in [0, T]\}$, where $\mathcal{F}_t = \sigma(W(s) : s \leq t)$ and the Brownian motion $W(t)$, $t \in [0, T]$ is a 1-dimensional process, defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}(\mathcal{F}), \mathbb{P})$, $t \in [0, T]$, where $\mathbb{P}$ is the real world probability measure and T the terminal time. σ is the volatility of stock with respect to changes in $W(t)$. $\mu > 0$ is the appreciation rate for stock. Moreover, σ is the volatility for the stock and is referred to as the coefficient of the market and is progressively measurable with respect to the filtration $\mathcal{F}$.

2.1. **Financial Models.**

In this paper, we assume that the pension fund administrator (PFA) faces a market that is characterized by a risk-free asset (cash account) and risky asset, all of whom are trade-able. Therefore, the dynamics of the underlying assets are given in (2.1) and (2.2):

$$dB(t) = rB(t)dt, B(0) = 1 \tag{2.1}$$

$$dS(t) = S(t)\left(\mu dt + \sigma dW(t)\right), S(0) = s_0 > 0 \tag{2.2}$$

where, r is the nominal interest rate, $B(t)$ is the price process of the cash account at time t, $S(t)$ is stock price process at time t.

We assume in this paper that the salary process $Y(t)$ at time t of the PPM satisfies

$$dY(t) = \beta Y(t)dt, Y(0) = y_0 > 0, \tag{2.3}$$

where β is the growth rate of the salary process of PPM. We assume that the PPM make the contributions because it is compulsory not because of the market conditions.

2.2. **The Wealth Dynamics.** Let $c > 0$ be the proportion of the PPM salary that is contributed into the pension plan (which is deducted at source), then the amount of contributions made by the PPM is $cY(t)$ at time t. Let $X(t)$ be the wealth process of the PPM at time t and $\Delta(t)$ be the portfolio process in stock at time t and $\Delta_0(t) = 1 - \Delta(t)$ is the proportion of wealth invested in cash account at time t. Therefore, the total wealth process defined as $X^y(t) = X(t) + cY(t)$ of the PPM is governs by the stochastic differential equation (SDE):

$$\begin{gathered} dX^y(t) = (X(t)(r + \Delta(t)(\mu - r)) + c\beta Y(t))\, dt + \Delta(t)X(t)\sigma dW(t), \\ X^y(0) = x_0^y = x_0 + cy_0 > 0 \end{gathered} \tag{2.4}$$

where, $X(t)$ satisfies the dynamics

$$\begin{gathered} dX(t) = X(t)(r + \Delta(t)(\mu - r))dt + \Delta(t)X(t)\sigma dW(t), \\ X(0) = x_0 \end{gathered} \tag{2.5}$$

The amount x_0 is the initial fund paid into PPM's account. If no amount is paid into the PPM account at the beginning, then the initial wealth is null. But, in this paper, we assume that at the beginning of the planning horizon, $x_0 > 0$ amount of money is paid into the PPM's account.

3. The Mean-Variance Approach

In this section, we assume that the PPM invests his/her contributions through the PFA from time 0 to time T. The aim of the PPM is to maximize his/her expected terminal wealth and simultaneously minimize the variance of the terminal wealth. Hence, the PPM aim at minimizing the vector

$$[-E(X(T)), Var(X(T))] \, .$$

Definition 1. *The portfolio strategy $\Delta(.)$ is said to be admissible if $\Delta(.) \in L^2_{\mathcal{F}}([0, T]; R)$.*

Definition 2. *The mean-variance optimization problem is defined as*

$$\min_{\Delta} J = [-E(X(T, \Delta)), Var(X(T, \Delta))] \tag{3.1}$$

subject to:

$$\begin{cases} \Delta(.), & \textit{set of admissible portfolio strategy} \\ X(.)\Delta(.), & \textit{satisfy}(2.5). \end{cases}$$

Solving (3.1) is equivalent to solving the following equation

$$\min_{\Delta} [-E(X(T, \Delta(.))) + \psi Var(X(T, \Delta(.)))], \psi > 0, \tag{3.2}$$

see [26]. [26] and [17] show that it is possible to transform (3.2) into a tractable one, see also [14]. They established that (3.2) is equivalent to the problem

$$\min_{\Delta(.)} E[\psi X^2(T) - \delta X(T)], \tag{3.3}$$

where,

$$\delta^* = 1 + 2\psi E(X^*(T)). \tag{3.4}$$

(3.4) can be re-express as in terms of $cE(Y(T))$ as follows:

$$\delta^* = 1 + 2\psi(E(\bar{X}^*(T)) - cy_0 e^{\beta T}), \tag{3.5}$$

where $X^y(t) = \bar{X}(t)$.

(3.3) is known as a linear-quadratic control problem. Hence, instead of solving (3.2), we now solve the following

$$\min(J(\Delta(.)), \psi, \delta) = E\left[\psi X(T, \Delta(.))^2 - \delta X(T, \Delta(.))\right], \tag{3.6}$$

subject to:

$$\begin{cases} \Delta(.), & \textit{set of admissible portfolio strategy} \\ X(.)\Delta(.), & \textit{satisfy}(2.5). \end{cases}$$

4. The Optimization problem

In solving (3.6), we set $\omega = \dfrac{\delta}{2\psi}$ and $H(t) = X(t) - \omega$, see [14] and [25]. It implies that

$$E\left[\psi X(t, \Delta(t))^2 - \delta X(t, \Delta(t))\right] = E\left[\psi(X(t)H(t) - X(t)^2)\right]. \tag{4.1}$$

Therefore, our problem is equivalent to solving

$$\min_{\Delta(.)} J\left(\Delta(.), \psi, \delta, y\right) = \min_{\Delta(.)} \left[\frac{\psi H(T)^2}{2}\right] \tag{4.2}$$

where the process $H(t)$ follows the SDE:

$$\begin{gathered} dH(t) = H(t) + \omega)\left(r + \Delta(t)(\mu - r)\right) dt + (H(t) + \omega)\Delta(t)\sigma dW(t), \\ H(0) = x - \omega = h \end{gathered} \tag{4.3}$$

$H(0) = x - \omega = h$, and

$$\begin{gathered} dH^y(t) = \left((H(t) + \omega)\left(r + \Delta(t)(\mu - r)\right) + c\beta Y(t)\right) dt + (H(t) + \omega)\Delta(t)\sigma dW(t), \\ H^y(0) = x - \omega + cy = h + cy \end{gathered} \tag{4.4}$$

(4.3) is a standard optimal stochastic control problem. Let

$$U(t, h, y) = \inf_{\Delta(.)} E_{t,h,y}\left[\frac{\psi H^y(T)^2}{2}\right] = \inf_{\Delta(.)} J\left(\Delta(.), \psi, \delta, y\right). \tag{4.5}$$

Then, the value function U satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{gathered} \inf_{\Delta \in R}\{U_t + (h + \omega)(\Delta(t)(\mu - r) + r)U_h + c\beta y U_y + \frac{1}{2}\sigma^2(h + \omega)^2\Delta_S^2(t)U_{hh}\} = 0, \\ U(T, h, y) = \frac{1}{2}\psi h^2 + cy \end{gathered} \tag{4.6}$$

Assuming U to be a convex function of h and y, then first order conditions lead to the optimal fraction of portfolios to be invested in stock at time t:

$$\Delta^*(t) = \frac{-(\mu - r)U_h}{\sigma^2(h + \omega)U_{hh}}. \tag{4.7}$$

Now, substituting (4.7) into (4.6), we have

$$U_t + r(h + \omega)U_h + c\beta y U_y - \frac{(\mu - r)^2 U_h^2}{2\sigma^2 U_{hh}} = 0. \tag{4.8}$$

4.1. **Optimization of a PPM's Portfolio and Wealth Using QUF.** In this paper, we assume the Quadratic Utility Function (QUF) of the form:

$$U(t,h,y) = P(t)h^2 + (Q(t)h + cR(t)y) + V(t). \tag{4.9}$$

Finding the partial derivative of U in (4.9) with respect to h, y and hh, and then substitute into (4.8), we have the following system of ordinary differential equations (ODEs):

$$P'(t) = (\theta^2 - 2r)P(t), \tag{4.10}$$

$$Q'(t) = (\theta^2 - r)Q(t) - 2r\omega P(t), \tag{4.11}$$

$$R'(t) = -c\beta R(t), \tag{4.12}$$

$$V'(t) = \frac{\theta^2 Q(t)^2}{4P(t)} - r\omega Q(t), \tag{4.13}$$

with boundary conditions
$P(T) = \frac{1}{2}\psi$, $Q(T) = 0$, $R(T) = 1$, $V(T) = 0$.
Solving the systems of ODEs in (4.10) to (4.13) using the boundary conditions, we have the following:

$$P(t) = \frac{1}{2}\psi e^{(2r-\theta^2)(T-t))} \tag{4.14}$$

$$Q(t) = \omega\psi e^{(-(\theta^2-r)(T-t))}[e^{((r(T-t))} - 1], \tag{4.15}$$

$$R(t) = e^{(c\beta(T-t))}, \tag{4.16}$$

$$V(t) = \int_T^t \left\{ \frac{\theta^2 Q(u)^2}{4P(u)} - r\omega Q(u) \right\} du \tag{4.17}$$

It then follows that by substituting (4.14)-(4.17) into (4.9), we have

$$\begin{aligned} U(t,h,y) &= \frac{h^2}{2}\psi e^{(2r-\theta^2)(T-t))} + \omega\psi h e^{(-(\theta^2-r)(T-t))}[e^{((r(T-t))} - 1] \\ &+ cye^{c\beta(T-t)} + \int_T^t \left\{ \frac{\theta^2 Q(u)^2}{4P(u)} - r\omega Q(u) \right\} du. \end{aligned}$$

This represents the utility of wealth the will accrued to the investor. At $t = 0$, we have

$$\begin{aligned} U(0, h = x_0 - \omega, y = y_0) &= \frac{(x_0-\omega)^2}{2}\psi e^{(2r-\theta^2)T)} + \omega\psi(x_0-\omega)e^{(-(\theta^2-r)T)}[e^{rT} - 1] \\ &+ cy_0 e^{c\beta T} - \int_0^T \left\{ \frac{\theta^2 Q(u)^2}{4P(u)} - r\omega Q(u) \right\} du. \end{aligned}$$

This represents the initial utility of wealth of the investor.

At $t = T$, we have

$$U(T,h,y) = \frac{h^2}{2}\psi + cy.$$

This represents the terminal utility of wealth of the investor.

We observe that our utility function U is indeed convex, since

$$U_{hh} = 2P(t) > 0, \psi > 0. \tag{4.18}$$

Now, substituting partial derivative of U into (4.7), we have the following:

$$\Delta^*(t) = \frac{-(\mu - r)}{\sigma^2(h+\omega)}[(h+\omega) - \omega e^{(-r(T-t))}]. \tag{4.19}$$

Hence, substituting $X^*(t)$ for $h+\omega$ and $\tau = t$ in (4.19), we have the following:

$$\Delta^*(t) = \frac{-(\mu - r)}{\sigma^2 X^*(t)}[X^*(t) - \omega e^{-r(T-t)}]. \tag{4.20}$$

The evolution of the stochastic fund for the PPM under optimal control $X^{y*}(t) = \bar{X}^*(t)$ can be obtained by substituting (4.20) into (2.5) as follows:

$$\begin{aligned} &d\bar{X}^*(t) = ((r-\theta^2)\bar{X}^*(t) + \omega\theta^2 e^{-r(T-t)} + c(\beta - r + \theta^2)Y(t))dt \\ &-\theta(\bar{X}^*(t) - cY(t) - \omega e^{-r(T-t)})dW(t) \end{aligned} \tag{4.21}$$

By application of Ito's formula to (4.21), we obtain the SDE that governs the evolution of $\bar{X}^{*2}(t)$:

$$\begin{aligned} &d\bar{X}^{*2}(t) = ((2r-\theta^2)\bar{X}^{*2}(t) + 2c(\beta - r)Y(t)\bar{X}^*(t) + 2cY(t)\omega^2\theta^2 e^{-r(T-t)} \\ &+c^2\theta^2 Y^2(t) + \theta^2\omega^2 e^{-2r(T-t)})dt - 2\theta(\bar{X}^{*2}(t) - cY(t)\bar{X}^*(t) - \bar{X}^*(t)\omega e^{-r(T-t)})dW(t). \end{aligned} \tag{4.22}$$

Taking the expectation on both sides of (4.21) and (4.22), we obtain the following ODEs:

$$\begin{cases} dE(\bar{X}^*(t)) = ((r-\theta^2)E(\bar{X}^*(t)) + \theta^2\omega e^{(-r(T-t))} + c(\beta - r + \theta^2)y_0 e^{\beta t})dt, \\ E(\bar{X}^*(0)) = \bar{x}_0 \end{cases} \tag{4.23}$$

$$\begin{cases} dE(\bar{X}^{*2}(t)) = ((2r-\theta^2)\bar{X}^{*2}(t) + 2c(\beta - r)y_0 e^{\beta t}\bar{X}^*(t) \\ +2cy_0 e^{\beta t}\omega^2\theta^2 e^{-r(T-t)} + c^2\theta^2 y_0^2 e^{2\beta t} + \theta^2\omega^2 e^{-2r(T-t)})dt \\ E(\bar{X}^{*2}(0)) = \bar{x}_0^2 \end{cases} \tag{4.24}$$

where $x_0^y = \bar{x}_0$. Solving the ODE (4.23) and (4.24), we find that the expected value of the wealth and its second moment under optimal control at time t are

$$E(\bar{X}^*(t)) = \bar{x}_0 e^{(r-\theta^2)t} + \omega e^{-r(T-t)}(1 - e^{-\theta^2 t}) + cy_0(e^{\beta t} - e^{(r-\theta^2)t}). \tag{4.25}$$

$$\begin{aligned} &E(\bar{X}^{*2}(t)) = \bar{x}_0^2 e^{(2r-\theta^2)t} + 2c(\beta - r)y_0[\frac{\bar{x}_0}{\beta - r}(e^{(\beta+r-\theta^2)t} - 1) \\ &+\omega\{(\frac{e^{-rT+(\beta+r)t}}{\beta - r + \theta^2} - \frac{e^{-rT}}{\beta - r + \theta^2}) - (\frac{e^{-rT+(\beta+r-\theta^2)t}}{\beta - r + \theta^2} - \frac{e^{-rT}}{\beta - r + \theta^2})\} \\ &+\frac{cy_0}{2\beta - 2r + \theta^2}((e^{2\beta t} - 1) - (\frac{e^{(\beta+r-\theta^2)t}}{\beta - r} - \frac{1}{\beta - r}))] + \omega^2(\frac{2cy_0\theta^2}{\beta - \theta^2 - r} \\ &(e^{-rT+(\beta+r)t} - e^{-rT}) + (e^{-2rT+\theta^2 t} - e^{-2rT})) + \frac{c^2\theta^2 y_0^2}{2\beta - 2r + \theta^2}(e^{2\beta t} - 1). \end{aligned} \tag{4.26}$$

At $t = T$, we have the expected terminal wealth of the PPM to be

$$E(\bar{X}^*(T)) = \bar{x}_0 e^{(r-\theta^2)T} + \omega(1 - e^{-\theta^2 T}) + cy_0(e^{\beta T} - e^{(r-\theta^2)T}). \tag{4.27}$$

At $t = T$, we have the second moment of expected terminal wealth of the PPM to be

$$\begin{aligned} &E(\bar{X}^{*2}(T)) = \bar{x}_0^2 e^{(2r-\theta^2)T} + 2c(\beta - r)y_0[\frac{\bar{x}_0}{\beta - r}(e^{(\beta+r-\theta^2)T} - 1) \\ &+\frac{cy_0(e^{2\beta T} - 1)}{2\beta - 2r + \theta^2} - \frac{cy_0}{\beta - r}(e^{(\beta+r-\theta^2)T} - 1)] + \frac{2c(\beta - r)y_0\omega}{\beta - r + \theta^2}(e^{\beta T} - e^{(\beta-\theta^2)T}) \\ &+\omega^2(\frac{2cy_0\theta^2}{\beta + \theta^2 - r}(e^{\beta T} - e^{-rT}) + (e^{-(2r-\theta^2)T} - e^{-2rT})) + \frac{c^2\theta^2 y_0^2}{2\beta - 2r + \theta^2}(e^{2\beta T} - 1). \end{aligned} \tag{4.28}$$

From (3.4) and (4.27) and the definition of ω^*, we have that ω^* is a decreasing function of ψ:

$$\omega^* = \frac{e^{\theta^2 T}}{2\psi} + (\bar{x}_0 - cy_0)e^{rT}. \tag{4.29}$$

Therefore, the expected optimal terminal wealth of the PPM can be rewritten in terms of ψ as follows:

$$E(\bar{X}^*(T)) = \bar{x}_0 e^{rT} + \frac{e^{\theta^2 T} - 1}{2\psi} + cy_0(1 - e^{rT}). \tag{4.30}$$

From (4.30), we observe that the expected terminal wealth of the PPM is the sum of the wealth that one would get investing the whole portfolio always in the risk-free asset plus the term, $cy_0(1 - e^{rT})$ that depend riskless asset and the initial contributions of the PPM, plus a term, $\frac{e^{\theta^2 T} - 1}{2\psi}$ that depend both on the goodness of the risky asset with respect to the risk-free one and on the weight given to the minimization of the variance. Thus, the higher the expected optimal terminal wealth value, for everything else being equal; the higher the variance minimization parameter, ψ, the lower its expected terminal wealth. In the same vain, the higher the growth rate of the contributions of PPM, the higher the terminal wealth of the PPM which is an intuitive result. We therefore conclude that the higher the *Sharpe ratio* θ^2 and the growth rate of salary of PPM, the higher the terminal wealth of the PPM.

Again, the expected second moment of optimal terminal wealth of the PPM can be rewritten as follows:

$$\begin{aligned}
&E(\bar{X}^{*2}(T)) = \bar{x}_0^2 e^{(2r-\theta^2)T} + 2c(\beta - r)y_0[\frac{\bar{x}_0}{\beta - r}(e^{(\beta+r-\theta^2)T} - 1)\\
&+\frac{cy_0(e^{2\beta T} - 1)}{2\beta - 2r + \theta^2} - \frac{cy_0}{\beta - r}(e^{(\beta+r-\theta^2)T} - 1)] + (\frac{1}{2\psi} - cy_0 e^{\beta T})\frac{2c(\beta - r)y_0}{(\beta - r + \theta^2)}(e^{\beta T} - e^{(\beta-\theta^2)T})\\
&+\frac{c^2\theta^2 y_0^2}{2\beta - 2r + \theta^2}(e^{2\beta T} - 1) + (\frac{1}{2\psi} - cy_0 e^{\beta T})^2(\frac{2cy_0\theta^2}{\beta + \theta^2 - r}(e^{\beta T} - e^{-rT}) + (e^{-(2r-\theta^2)T} - e^{-2rT}))\\
&+E(\bar{X}^*(T))[\frac{2c(\beta - r)y_0}{\beta - r + \theta^2}(e^{\beta T} - e^{(\beta-\theta^2)T}) + (\frac{1}{2\psi} - cy_0 e^{\beta T})(\frac{2cy_0\theta^2}{\beta + \theta^2 - r}(e^{\beta T} - e^{-rT})\\
&+(e^{-(2r-\theta^2)T} - e^{-2rT}))] + (E(\bar{X}^*(T)))^2(\frac{2cy_0\theta^2}{\beta + \theta^2 - r}(e^{\beta T} - e^{-rT})\\
&+(e^{-(2r-\theta^2)T} - e^{-2rT})).
\end{aligned} \tag{4.31}$$

The optimal proportion of wealth to be invested in risky asset can be expressed in terms of ψ and terminal wealth as follows:

$$\Delta^*(t) = \frac{(\mu - r)}{\gamma\sigma_S\sigma_S' X^*(t)}[X^*(t) - E(\bar{X}^*(T))e^{-r(T-t)} + cy_0 e^{\beta T - r(T-t)} - \frac{e^{-r(T-t)}}{2\psi}]. \tag{4.32}$$

At $t = 0$, we have

$$\Delta^*(0) = \frac{(\mu - r)}{\gamma\sigma_S\sigma_S' x_0}[x_0 - E(\bar{X}^*(T))e^{-rT} + cy_0 e^{(\beta-r)T} - \frac{e^{-rT}}{2\psi}]. \tag{4.33}$$

From (4.32), the amount $X^*(t)\Delta^*(t)$ invested in risky asset at time t is proportional to the difference between the wealth $X^*(t)$ at time t and the wealth available investing always in cash account, minus the wealth available investing in risk-free asset (i.e., cash account), minus a term that depends on ψ, r and the time to retirement. In addition, the higher the weight of the variance minimizer , the higher the amount invested in the risky asset and vice versa. Hence, the strategy of investing the portfolio in both risk-free and stock is optimal if and only if the weight ψ, tends to infinity. Therefore, the amount $X^*(t)\Delta^*(t)$ invested in stock at time t is proportional to the difference between the wealth $X^*(t)$ at time t and the amount that would be sufficient to guarantee the attainment of the target by the adoption of both the cash account and stock strategies until retirement.

If $\psi \to \infty$, the expected portfolio of the PPM will be

$$\Delta^*(t) = -\frac{(\mu - r)}{X^*(t)\sigma^2}\left(X^*(t) - (E(\bar{X}^*(T))e^{-r(T-t)} + cy_0 e^{\beta T - r(T-t)}\right). \tag{4.34}$$

(4.34) shows that the expected terminal wealth of a PPM will increase if the underlying assets are less volatile, which is an expected result. Again, if $\psi \to \infty$, the expected terminal wealth of the PPM will be

$$E(\bar{X}^*(T)) = \bar{x}_0 e^{rT} + cy_0(1 - e^{rT}). \tag{4.35}$$

4.2. **Efficient Frontier under QUF.** The variance of the optimal portfolio under QUF is given as follows:

$$\begin{aligned}
&Var(\bar{X}^*(T)) = E(\bar{X}^{*2}(T)) - [E(\bar{X}^*(T))]^2 = \bar{x}_0^2 e^{(2r-\theta^2)T} + 2c(\beta - r)y_0[\frac{\bar{x}_0}{\beta - r}(e^{(\beta+r-\theta^2)T} - 1)\\
&+\frac{cy_0(e^{2\beta T} - 1)}{2\beta - 2r + \theta^2} - \frac{cy_0}{\beta - r}(e^{(\beta+r-\theta^2)T} - 1)] + (\frac{1}{2\psi} - cy_0 e^{\beta T})\frac{2c(\beta - r)y_0}{(\beta - r + \theta^2)}(e^{\beta T} - e^{(\beta-\theta^2)T})\\
&+\frac{c^2\theta^2 y_0^2}{2\beta - 2r + \theta^2}(e^{2\beta T} - 1) + (\frac{1}{2\psi} - cy_0 e^{\beta T})^2(\frac{2cy_0\theta^2}{\beta + \theta^2 - r}(e^{\beta T} - e^{-rT}) + (e^{-(2r-\theta^2)T} - e^{-2rT}))\\
&+E(\bar{X}^*(T))[\frac{2c(\beta - r)y_0}{\beta - r + \theta^2}(e^{\beta T} - e^{(\beta-\theta^2)T}) + (\frac{1}{2\psi} - cy_0 e^{\beta T})(\frac{2cy_0\theta^2}{\beta + \theta^2 - r}(e^{\beta T} - e^{-rT})\\
&+(e^{-(2r-\theta^2)T} - e^{-2rT}))] + (E(\bar{X}^*(T)))^2(\frac{2cy_0\theta^2}{\beta + \theta^2 - r}(e^{\beta T} - e^{-rT})\\
&+(e^{-(2r-\theta^2)T} - e^{-2rT})).
\end{aligned} \tag{4.36}$$

Setting

$$\begin{aligned}
&z_1(T) = \bar{x}_0^2 e^{(2r-\theta^2)T} + 2c(\beta - r)y_0[\frac{\bar{x}_0}{\beta - r}(e^{(\beta+r-\theta^2)T} - 1)\\
&+\frac{cy_0(e^{2\beta T} - 1)}{2\beta - 2r + \theta^2} - \frac{cy_0}{\beta - r}(e^{(\beta+r-\theta^2)T} - 1)] + (\frac{1}{2\psi} - cy_0 e^{\beta T})\frac{2c(\beta - r)y_0}{(\beta - r + \theta^2)}(e^{\beta T} - e^{(\beta-\theta^2)T})\\
&+\frac{c^2\theta^2 y_0^2}{2\beta - 2r + \theta^2}(e^{2\beta T} - 1) + (\frac{1}{2\psi} - cy_0 e^{\beta T})^2(\frac{2cy_0\theta^2}{\beta + \theta^2 - r}(e^{\beta T} - e^{-rT}) + (e^{-(2r-\theta^2)T} - e^{-2rT})),
\end{aligned}$$

$$z_2(T) = \frac{2c(\beta - r)y_0}{\beta - r + \theta^2}(e^{\beta T} - e^{(\beta-\theta^2)T}) + (\frac{1}{2\psi} - cy_0 e^{\beta T})(\frac{2cy_0\theta^2}{\beta + \theta^2 - r}(e^{\beta T} - e^{-rT}) + (e^{-(2r-\theta^2)T} - e^{-2rT})),$$

and

$$z_3(T) = (\frac{2cy_0\theta^2}{\beta + \theta^2 - r}(e^{\beta T} - e^{-rT}) + (e^{-(2r-\theta^2)T} - e^{-2rT})) - 1,$$

then (4.36) becomes

$$Var(\bar{X}^*(T)) = z_1(T) + z_2(T)E(\bar{X}^*(T)) + z_3(T)(E(\bar{X}^*(T)))^2. \tag{4.37}$$

It implies that the standard deviation of the terminal wealth will be

$$\sigma(\bar{X}^*(T)) = \sqrt{z_1(T) + z_2(T)E(\bar{X}^*(T)) + z_3(T)(E(\bar{X}^*(T)))^2}. \tag{4.38}$$

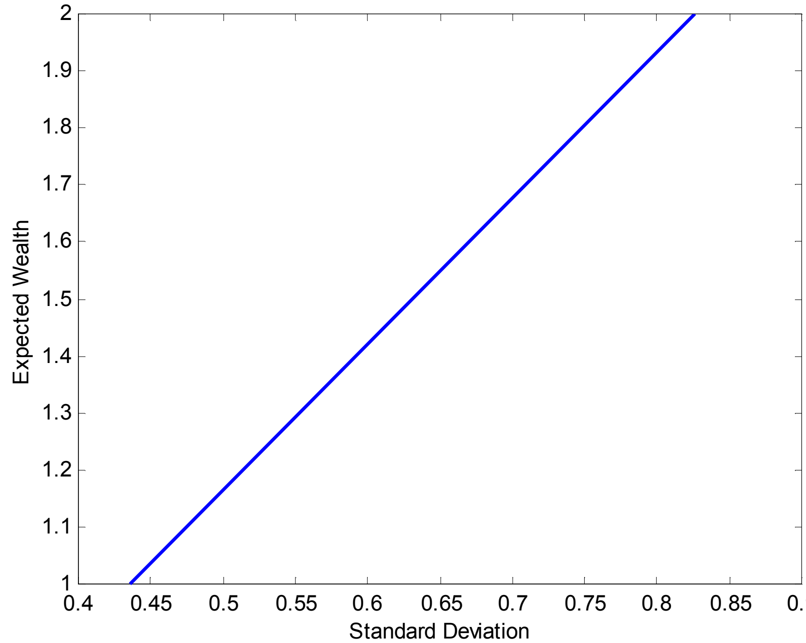

FIGURE 1. Efficient Frontier under QUF. This is obtained by setting $\mu = 0.09$, $\sigma = 0.3$, $\beta = 0.0292$, $c = 0.15$, $y_0 = 0.9$, $T = 20$, $r = 0.04$, $x_0 = 0.865$, $\psi = 1$.

If $Var(\bar{X}^*(T)) = 0$, it implies that $z_3(T)(E(\bar{X}^*(T)))^2 + z_2(T)E(\bar{X}^*(T)) + z_1(T) = 0$. It then follows that

$$E(\bar{X}^*(T)) = \frac{-z_2(T) \pm \sqrt{z_2(T)^2 - 4z_1(T)z_3(T)}}{2z_3(T)}. \tag{4.39}$$

Observe from (4.39) that if $z_2(T)^2 \geq 4z_1(T)z_3(T)$, the expected terminal wealth will have real values, and real and imaginary values when $z_2(T)^2 < 4z_1(T)z_3(T)$. (4.39) gives the expected terminal wealth of the PPM when the portfolio is free from risky under QUF.

We found from figure 3 that $\lim_{\bar{X}^*(t) \longrightarrow +\infty} \Delta^*(t) = -0.55556$ and from figure 4 that $\lim_{\bar{X}^*(t) \longrightarrow +\infty} \Delta_0^*(t) = 1.55556$.

Figure 1 shows the efficient frontier of the two classes of assets under QUF. It shows that for a wealth of 1 to 2 million, the PPM stand the risk of losing 0.0 to 0.9 million. Figure 2 shows the efficient frontier under QUF but with the wealth 1 to 6 million and stands the risk of losing 0 to 3.5 million. Figure 3 shows the portfolio value of the PPM in a riskless asset up to retirement. It is observe that the portfolio value in riskless asset is nonnegative overtime. Figure 4 shows the portfolio value of a PPM in risky asset under QUF. It is observe that as time goes on, the portfolio value in risky asset will tends to a negative value. Therefore, the portfolio in the risky asset should be gradually move to the riskless asset.

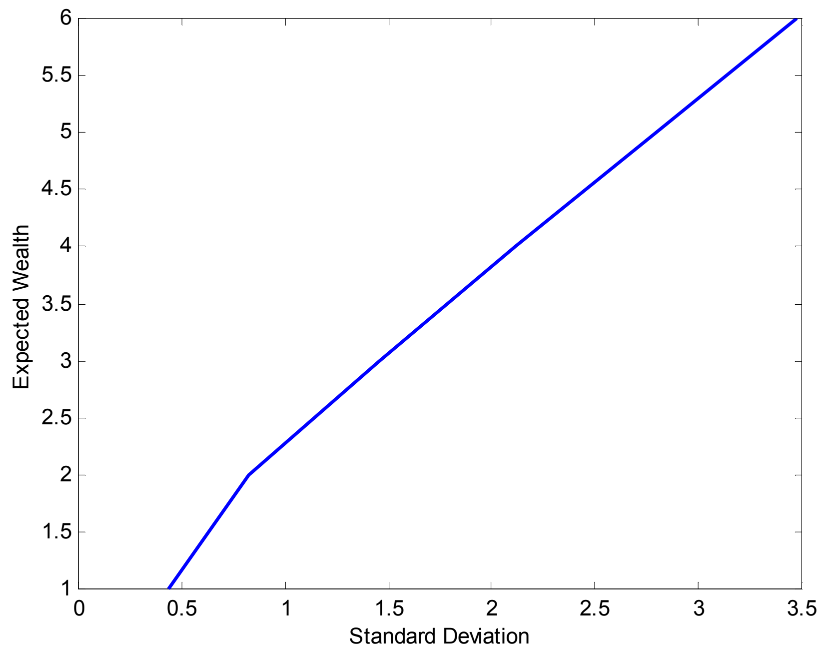

FIGURE 2. Efficient Frontier under QUF. This is obtained by setting $\mu = 0.09$, $\sigma = 0.3$, $\beta = 0.0292$, $c = 0.15$, $y_0 = 0.9$, $T = 20$, $r = 0.04$, $x_0 = 0.865$, $\psi = 1$.

5. Optimization of a PPM's Portfolio and Wealth Using EUF

Consider the EUF

$$U_e(t,h,y) = -e^{-\alpha(hQ_e(t)+cyP_e(t))+R(t)}.$$

with (constant) Arrow-Pratt coefficient of absolute risk aversion equal to

$$ARA(h) = -\frac{U_e''(h)}{U_e'(h)} = \alpha > 0.$$

Finding the partial derivative of $U_e(t,h,y)$ with respect to t, h, y and hh and substitute into (4.8), we have the following ODEs:

$$\begin{cases} Q_e'(t) + rQ_e(t) = 0 \\ P_e'(t) + cP_e(t) = 0 \\ R_e'(t) - r\omega_e\alpha Q_e(t) - \dfrac{\theta^2}{2} = 0 \end{cases} \tag{5.1}$$

Solving the ODEs (5.1), we have

$$\begin{cases} Q_e(t) = e^{r(T-t)} \\ Q_e(T) = 1 \end{cases} \tag{5.2}$$

$$\begin{cases} P_e(t) = e^{c(T-t)} \\ P_e(T) = 1 \end{cases} \tag{5.3}$$

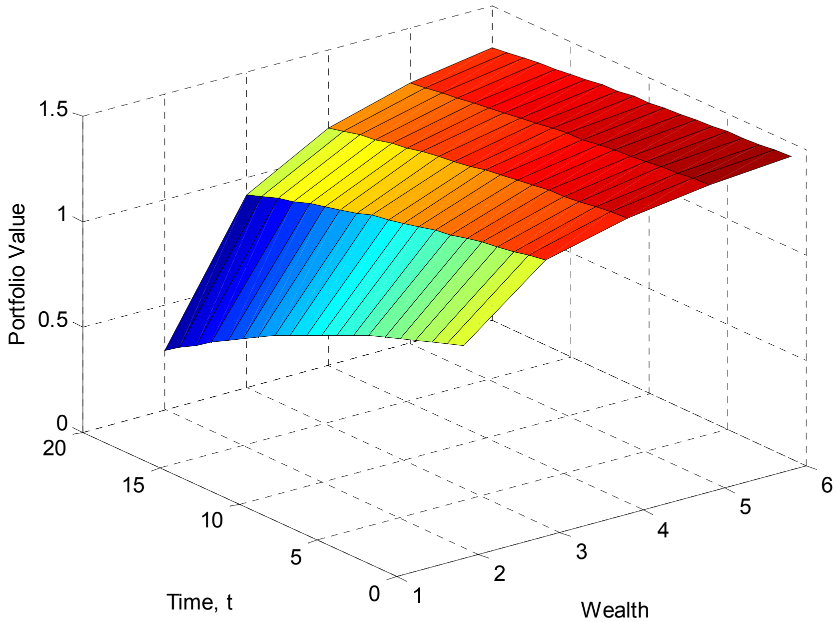

FIGURE 3. Portfolio Value in Riskless Asset under QUF. This is obtained by setting $\mu = 0.09$, $\sigma = 0.3$, $\beta = 0.0292$, $c = 0.15$, $y_0 = 0.9$, $T = 20$, $r = 0.04$, $x_0 = 0.865$, $\psi = 1$.

$$\begin{cases} R_e(t) = \alpha\omega(e^{r(T-t)} - 1) - \dfrac{\theta^2(T-t)}{2}, \\ R_e(T) = 0 \end{cases} \tag{5.4}$$

The utility function under this strategy now becomes

$$\begin{aligned} &U_e(t, h, y) = -e^{-\alpha(he^{r(T-t)}+cye^{c(T-t)})+\alpha\omega(e^{r(T-t)}-1)-\frac{\theta^2(T-t)}{2}}, \\ &U_e(T, h, y) = -e^{-\alpha(h+cy)}. \end{aligned}$$

At $t = 0$, we have

$$U_e(0, h, y) = -e^{-\alpha(he^{rT}+cye^{cT})+\alpha\omega(e^{rT}-1)-\frac{\theta^2 T}{2}}.$$

Replacing partial derivatives of U (4.8) and replacing $h + \omega_e$ with $X_e^*(t)$ yields

$$\Delta_e^*(t) = \frac{(\mu - r)e^{-r(T-t)}}{\alpha\sigma^2 X_e^*(t)}. \tag{5.5}$$

Observe that when the optimal fund is considerably large, it implies that the entire fund should remain only in cash account until retirement. The evolution of the fund under optimal control $\bar{X}_e^*(t)$ can be easily obtained as

$$d\bar{X}_e^*(t) = \left(r\bar{X}_e^*(t) + \frac{\theta^2}{\alpha}e^{-r(T-t)} + c(\beta - r)Y(t)\right)dt + \frac{\theta^2}{\alpha}e^{-r(T-t)}dW(t). \tag{5.6}$$

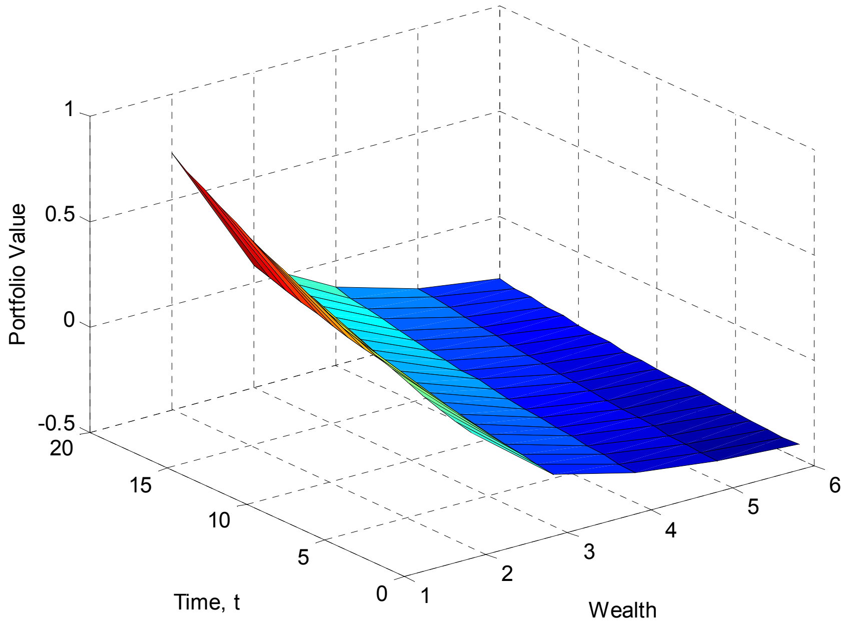

FIGURE 4. Portfolio Value in Risky Asset under QUF. This is obtained by setting $\mu = 0.09$, $\sigma = 0.3$, $\beta = 0.0292$, $c = 0.15$, $y_0 = 0.9$, $T = 20$, $r = 0.04$, $x_0 = 0.865$, $\psi = 1$.

By application of *Itô* formula to (5.6), we obtain

$$\begin{aligned} d\bar{X}_e^{*2}(t) &= (2rX_e^{*2}(t) + (\frac{2\theta^2}{\alpha}e^{-r(T-t)} + 2c(\beta - r)Y(t))\bar{X}_e^*(t) \\ &+\frac{\theta^4}{\alpha^2}e^{-2r(T-t)})dt + \frac{2\theta^2}{\alpha}e^{-2r(T-t)}dW(t) \end{aligned} \tag{5.7}$$

Taking the expectation of both sides of (5.6) and (5.7), we find that the expected value of the optimal fund and the expected value of its square (i.e. the moment) follow the linear ODEs:

$$\begin{cases} dE(\bar{X}_e^*(t)) = \left(rE(\bar{X}_e^*(t)) + \dfrac{\theta^2}{\alpha}e^{-r(T-t)} + c(\beta - r)y_0e^{\beta t}\right)dt \\ E(\bar{X}_e^*(0)) = \bar{x}_0 \end{cases} \tag{5.8}$$

$$\begin{cases} dE(\bar{X}_e^{*2}(t)) = (2rE(X_e^{*2}(t)) + (\dfrac{2\theta^2}{\alpha}e^{-r(T-t)} \\ +2c(\beta - r)y_0e^{\beta t})E(\bar{X}_e^*(t)) + \dfrac{\theta^4}{\alpha^2}e^{-2r(T-t)})dt \\ E(X_e^{*2}(0)) = \bar{x}_0^2 \end{cases} \tag{5.9}$$

By solving the ODEs, we find that the expected value of the fund under optimal control at time t under EUF is:

$$E(\bar{X}_e^*(t)) = \bar{x}_0e^{rt} - cy_0\left(e^{rt} - e^{\beta t}\right) + \frac{\theta^2 t}{\alpha}e^{-r(T-t)} \tag{5.10}$$

and the expected value of the square of the fund under optimal control at time t under EUF is:

$$E(\bar{X}_e^{*2}(t)) = \left(\frac{c\beta y_0 e^{\beta t} - (r\bar{x}_0 + c\beta y_0 - \bar{x}_0\beta)e^{rt}}{r-\beta}\right)^2 - \frac{2t\theta^2}{\alpha(r+\beta)}\left(c\beta y_0 e^{-r(T-t)+\beta t} - (r\bar{x}_0 + c\beta y_0 - \bar{x}_0\beta)e^{2rt-rT}\right) + \frac{t\theta^2(\alpha + t\theta^2)}{\alpha^2}e^{-2(T-t)} \tag{5.11}$$

At terminal time T, we have:

$$E(\bar{X}_e^*(T)) = \bar{x}_0 e^{rT} - cy_0\left(e^{rT} - e^{\beta T}\right) + \frac{\theta^2 T}{\alpha} \tag{5.12}$$

and

$$E(\bar{X}_e^{*2}(T)) = \left(\frac{c\beta y_0 e^{\beta T} - (r\bar{x}_0 + c\beta y_0 - \bar{x}_0\beta)e^{rT}}{r-\beta}\right)^2 - \frac{2T\theta^2}{\alpha(r+\beta)}\left(c\beta y_0 + \beta T) - (r\bar{x}_0 + c\beta y_0 - \bar{x}_0\beta)e^{rT}\right) + \frac{T\theta^2(\alpha + T\theta^2)}{\alpha^2} \tag{5.13}$$

(5.13) is equivalent to:

$$E(\bar{X}_e^{*2}(T)) = (E(\bar{X}_e^*(T))^2 + \frac{T\theta^2}{\alpha} \tag{5.14}$$

Therefore, the variance of the final fund under EUF is obtain as:

$$Var(\bar{X}_e^*(T)) = E(\bar{X}_e^{*2}(T)) - (E(\bar{X}_e^*(T))^2 = \frac{T\theta^2}{\alpha}. \tag{5.15}$$

This result coincides with that obtained by [25] under constant flow of contribution of a PPM. This then lead to the following proposition.

Proposition 5.1. *Let $E(\bar{X}_e^*(T))$ be the expected terminal fund of a PPM with time-dependent salary under EUF and $E(\bar{X}_c^*(T))$ the expected terminal fund with constant flow of contribution (for detail, see* [25]*), then*
(i) $E(\bar{X}_e^*(T)) \neq E(\bar{X}_c^*(T))$;
(ii) $E(\bar{X}_e^*(T)) = E(\bar{X}_c^*(T))$ if and only if $y = 1$ and $\beta = 0$; and
(iii) $Var(\bar{X}_e^*(T)) = Var(\bar{X}_c^*(T))$.

Figure 5 shows the portfolio value in risky asset under EUF. It is observe that the portfolio value will remain nonnegative overtime. The portfolio value in a riskless asset under EUF is shown in figure 6. It is also observe that the portfolio value is nonnegative at retirement.

6. Optimization of a PPM's Portfolio and Wealth Using PUF

Consider the following PUF

$$U_p(h, y) = \frac{(h+cy)^{1-\gamma}}{1-\gamma}, \gamma < 1. \tag{6.1}$$

In this section therefore, we adopt the utility function defined in (6.1). Then, the optimal portfolio value of the PPM under PUF is obtained as:

$$\Delta_P^*(t) = \frac{(\mu - r)(h + cy)}{\gamma\sigma^2(h + \omega_p)} \tag{6.2}$$

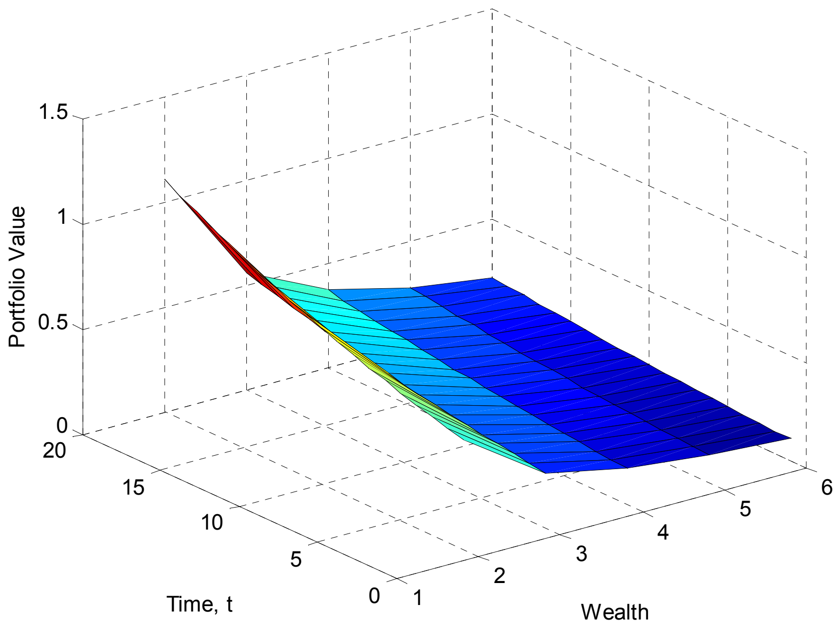

FIGURE 5. Portfolio Value in Risky Asset under EUF: This is obtained by setting $\mu = 0.09$, $\sigma = 0.3$, $\beta = 0.0292$, $c = 0.15$, $y_0 = 0.9$, $T = 20$, $r = 0.04$, $\alpha = 0.5$, $\psi = 1$, $x_0 = 0.865$.

Equivalently,

$$\Delta_P^*(t) = \frac{(\mu - r)(X_P^*(t) + cY(t) - \omega_p)}{\gamma\sigma^2 \bar{X}_P^*(t)} \tag{6.3}$$

where, $\bar{X}_P^*(t)$ is the wealth process of the PPM at time t under PUF. Substituting (6.3) into (2.5), we have

$$\begin{aligned} d\bar{X}_P^*(t) &= \left((r + \frac{\theta^2}{\gamma})\bar{X}_P^*(t) - \frac{\theta^2 \omega_p}{\gamma} + c(\beta - r)Y(t) \right) dt \\ &+ \frac{\theta}{\gamma} \left(\bar{X}_P^*(t) - \omega_p \right) dW(t) \end{aligned} \tag{6.4}$$

By application of $It\hat{o}$ lemma to (6.4), we obtain the SDE that governs the evolution of $\bar{X}_P^{*2}(t)$:

$$\begin{aligned} d\bar{X}_P^{*2}(t) &= ((2r + \frac{2\theta^2}{\gamma} + \frac{\theta^2}{\gamma^2})\bar{X}_P^{*2}(t) + (2c(\beta - r)Y(t) - \frac{2\theta^2\omega_p}{\gamma^2}(\frac{\theta^2}{\gamma} + 1))\bar{X}^*(t) + \frac{\theta^2\omega_p^2}{\gamma^2})dt \\ &+ \frac{2\theta\bar{X}_P^*(t)}{\gamma}(\bar{X}_P^*(t) - \omega_p)dW(t) \end{aligned} \tag{6.5}$$

Taking mathematical expectation of (6.4) and (6.5), we obtain the following ODEs:

$$\begin{cases} dE(\bar{X}_P^*(t)) = \left((r + \dfrac{\theta^2}{\gamma})E(\bar{X}_P^*(t)) - \dfrac{\theta^2\omega_p}{\gamma} + c(\beta - r)Y(t) \right) dt \\ E(\bar{X}_P^*(0)) = \bar{x}_0 \end{cases} \tag{6.6}$$

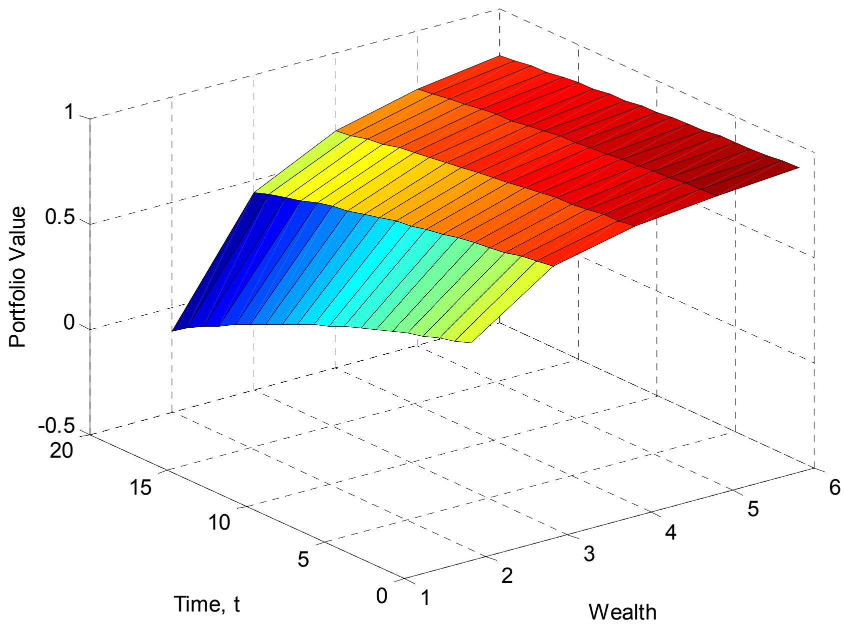

FIGURE 6. Portfolio Value in Riskless Asset under EUF: This is obtained by setting $\mu = 0.09$, $\sigma = 0.3$, $\beta = 0.0292$, $c = 0.15$, $y_0 = 0.9$, $T = 20$, $r = 0.04$, $\alpha = 0.5$, $\psi = 1$, $x_0 = 0.865$.

$$\left\{ \begin{array}{l} dE(\bar{X}_P^{*2}(t)) = ((2r + \frac{2\theta^2}{\gamma} + \frac{\theta^2}{\gamma^2})E(\bar{X}_P^{*2}(t)) + (2c(\beta - r)Y(t) \\ -\frac{2\theta^2\omega_p}{\gamma^2}(\frac{\theta^2}{\gamma} + 1))E(\bar{X}^*(t)) + \frac{\theta^2\omega_p^2}{\gamma^2})dt, \\ E(\bar{X}_P^{*2}(0)) = \bar{x}_0^2. \end{array} \right. \tag{6.7}$$

Then, the expected wealth of the PPM at time t is

$$E(\bar{X}_P^*(t)) = \bar{x}_0 e^{(r+\frac{\theta^2}{\gamma})t} + \frac{\gamma\theta^2\omega_p(1 - e^{(r+\frac{\theta^2}{\gamma})t})}{r\gamma + \theta^2} + \frac{\gamma c(\beta - r)y_0}{\beta\gamma - r\gamma - \theta^2}(e^{\beta t} - e^{(r+\frac{\theta^2}{\gamma})t}). \tag{6.8}$$

At $t = T$, (6.8) becomes

$$E(\bar{X}_P^*(T)) = \bar{x}_0 e^{(r+\frac{\theta^2}{\gamma})T} + \frac{\gamma\theta^2\omega_p(1 - e^{(r+\frac{\theta^2}{\gamma})T})}{r\gamma + \theta^2} + \frac{\gamma c(\beta - r)y_0}{\beta\gamma - r\gamma - \theta^2}(e^{\beta T} - e^{(r+\frac{\theta^2}{\gamma})T}). \tag{6.9}$$

But, $\omega_p^* = \frac{1}{2\psi} + E(\bar{X}_P^*(T)) - cy_0 e^{\beta T}$. It implies that

$$\omega_p^* = \frac{r\gamma + \theta^2}{r\gamma + \theta^2 - \gamma\theta^2(1 - e^{(r+\frac{\theta^2}{\gamma})T})}[\frac{1}{2\psi} - cy_0 e^{\beta T} + \bar{x}_0 e^{(r+\frac{\theta^2}{\gamma})T}$$
$$+ \frac{\gamma c(\beta - r)y_0}{\beta\gamma - r\gamma - \theta^2}(e^{\beta T} - e^{(r+\frac{\theta^2}{\gamma})T})].$$

By definition of ω_p, we observe that ω_p is decreasing function of ψ.
The expected optimal final fund can be rewritten in terms of ψ:

$$\begin{aligned} E(\bar{X}_P^*(T)) = &\frac{r\gamma+\theta^2}{r\gamma+\theta^2-\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}[\bar{x}_0 e^{(r+\frac{\theta^2}{\gamma})T} + \frac{\gamma c(\beta-r)y_0}{\beta\gamma-r\gamma-\theta^2}(e^{\beta T}-e^{(r+\frac{\theta^2}{\gamma})T}) \\ &-\frac{cy_0 e^{\beta T}\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}{r\gamma+\theta^2} + \frac{\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}{2\psi(r\gamma+\theta^2)}]. \end{aligned} \tag{6.10}$$

It is easy to see that the expected optimal final fund is the sum of the fund that one would get for investing the whole portfolio always in the risky asset with respect to the riskless one plus the term $\frac{r\gamma+\theta^2}{r\gamma+\theta^2-\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}(\frac{\gamma c(\beta-r)y_0}{\beta\gamma-r\gamma-\theta^2}(e^{\beta T} - e^{(r+\frac{\theta^2}{\gamma})T}) - \frac{cy_0e^{\beta T}\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}{r\gamma+\theta^2})$ that depend on the contribution, γ, r, β and Sharpe ratio. It is additional returns to the final fund generated from the contributions of the PPM plus a term $\frac{\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}{2\psi(r\gamma+\theta^2-\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T}))}$ that depends both on the goodness of the risky asset with respect to the riskless one and the weight given to the minimization of the variance. Hence, the higher the *Sharpe ratio*, θ, the higher the expected optimal final fund of the PPM, everything else being equal. Observe that as $\psi \to \infty$, the expected terminal fund of the PPM decreases (i.e., the higher the importance given to the minimization of the variance of the final fund, ψ, the lower its expected terminal fund). In that case, (6.10) becomes

$$\begin{aligned} E(\bar{X}_P^*(T))|_{\psi=\infty} = &\frac{r\gamma+\theta^2}{r\gamma+\theta^2-\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}[\bar{x}_0 e^{(r+\frac{\theta^2}{\gamma})T} \\ &+\frac{\gamma c(\beta-r)y_0}{\beta\gamma-r\gamma-\theta^2}(e^{\beta T}-e^{(r+\frac{\theta^2}{\gamma})T}) - \frac{cy_0 e^{\beta T}\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}{r\gamma+\theta^2}]. \end{aligned} \tag{6.11}$$

In the same vain, if $y=0$, we have

$$E(\bar{X}_P^*(T)) = \frac{r\gamma+\theta^2}{r\gamma+\theta^2-\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}[\bar{x}_0 e^{(r+\frac{\theta^2}{\gamma})T} + \frac{\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}{2\psi(r\gamma+\theta^2)}]. \tag{6.12}$$

We now express the optimal portfolio under PUF as a function of variance minimizer and expected final wealth as follows:

$$\Delta_P^*(t) = \frac{(\mu-r)}{\gamma\sigma^2 X_P^*(t)}\left(X_P^*(t) - E(\bar{X}_P^*(T)) + cy_0(e^{\beta t}-e^{\beta T}) - \frac{1}{2\psi}\right) \tag{6.13}$$

If we allow $\psi \to \infty$, (6.13) becomes

$$\Delta_P^*(t) = \frac{(\mu-r)}{\gamma\sigma^2 X_P^*(t)}\left(X_P^*(t) - E(\bar{X}_P^*(T)) + cy_0(e^{\beta t}-e^{\beta T})\right). \tag{6.14}$$

It is observe that the portfolio value of the PPM increases as ψ becomes large.

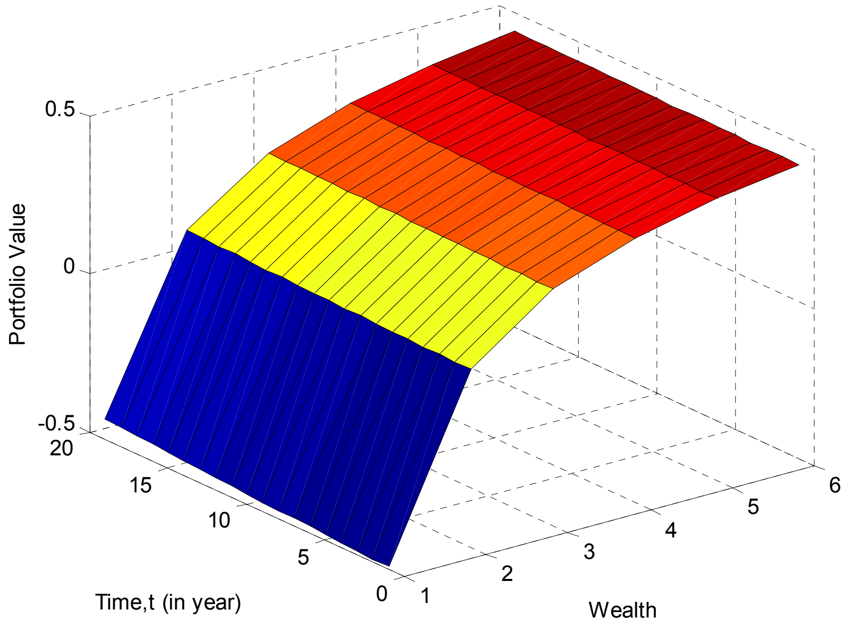

FIGURE 7. Porfolio Value in stock for a PPM under PUF. This is obtained by setting $\mu = 0.09$, $\sigma = 0.3$, $\beta = 0.0292$, $c = 0.15$, $y_0 = 0.9$, $T = 20$, $r = 0.04$, $\gamma = 0.9$, $\psi = 1$, $x_0 = 0.865$.

The expected value of the square of the fund under optimal control at time t is:

$$
\begin{aligned}
&E(\bar{X}_P^{*2}(t)) = \bar{x}_0^2 e^{\nu t} + \left(\frac{k\bar{x}_0}{\beta-\nu+z} - \frac{kc(\beta-r)y_0}{(\beta-z)(\beta-\nu+z)}\right)(e^{(\beta+z)t} - e^{\nu t}) \\
&+\frac{kc(\beta-r)y_0}{(\beta-z)(2\beta-\nu)}(e^{2\beta t} - e^{\nu t}) + w_p^*\{\left(\frac{k\theta^2}{z(\beta-\nu)} - \frac{gc(\beta-r)y_0}{(\beta-z)(\beta-\nu)}\right)(e^{\beta t} - e^{\nu t}) \\
&+\left(\frac{gc(\beta-r)y_0}{(\beta-z)(z-\nu)} - \frac{g\bar{x}_0}{z-\nu}\right)(e^{zt} - e^{\nu t}) - \frac{k\theta^2}{z(\beta-\nu+z)}(e^{(\beta+z)t} - e^{\nu t})\} \\
&+\omega_p^{*2}\{\left(\frac{\theta^2}{z\nu} - \frac{\theta^2}{\gamma^2\nu}\right)(1 - e^{\nu t}) + \frac{g\theta^2}{z(z-\nu)}(e^{zt} - e^{\nu t})\}
\end{aligned}
\tag{6.15}
$$

where,

$k = 2c^2 y_0(\beta - r)$, $z = r + \dfrac{\theta^2}{\gamma}$, $g = \dfrac{2\theta^2}{\gamma^2}\left(\dfrac{\theta^2}{\gamma} + 1\right)$ and $\nu = 2z + \dfrac{\theta^2}{\gamma^2}$.

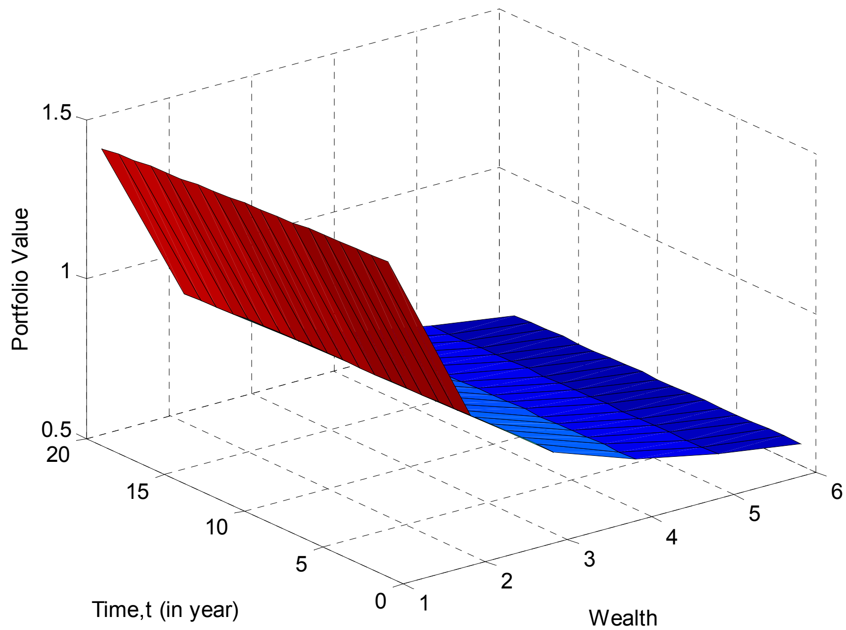

FIGURE 8. Porfolio Value in Cash Account for a PPM under PUF. This is obtained by setting $\mu = 0.09$, $\sigma = 0.3$, $\beta = 0.0292$, $c = 0.15$, $y_0 = 0.9$, $T = 20$, $r = 0.04$, $\gamma = 0.9$, $\psi = 1$, $x_0 = 0.865$.

At $t = T$, we have

$$
\begin{aligned}
&E(\bar{X}_P^{*2}(T)) = \bar{x}_0^2 e^{\nu T} + \left(\frac{k\bar{x}_0}{\beta - \nu + z} - \frac{kc(\beta - r)y_0}{(\beta - z)(\beta - \nu + z)} \right) (e^{(\beta+z)T} - e^{\nu T}) \\
&+ \frac{kc(\beta - r)y_0}{(\beta - z)(2\beta - \nu)} (e^{2\beta T} - e^{\nu T}) + w_p^* \{ \left(\frac{k\theta^2}{z(\beta - \nu)} - \frac{gc(\beta - r)y_0}{(\beta - z)(\beta - \nu)} \right) (e^{\beta T} - e^{\nu T}) \\
&+ \left(\frac{gc(\beta - r)y_0}{(\beta - z)(z - \nu)} - \frac{g\bar{x}_0}{z - \nu} \right) (e^{zT} - e^{\nu T}) - \frac{k\theta^2}{z(\beta - \nu + z)} (e^{(\beta+z)T} - e^{\nu T}) \} \\
&+ \omega_p^{*2} \{ \left(\frac{\theta^2}{z\nu} - \frac{\theta^2}{\gamma^2 \nu} \right) (1 - e^{\nu T}) + \frac{g\theta^2}{z(z - \nu)} (e^{zT} - e^{\nu T}) \}.
\end{aligned}
\tag{6.16}
$$

6.1. **Efficient Frontier under PUF.** We now express the second moment of the optimal wealth as a function of ψ as follows:

$$
\begin{aligned}
E(\bar{X}_P^{*2}(T)) &= K_1 + \omega_p^* K_2 + \omega_p^{*2} K_3 \\
&= K_1 + (\frac{1}{2\psi} - cy_0 e^{\beta t}) K_2 + (\frac{1}{2\psi} - cy_0 e^{\beta T})^2 K_3 + E(X^*(T))(K_2 \\
&+ 2(\frac{1}{2\psi} - cy_0 e^{\beta T}) K_3) + E(X^*(T))^2 K_3
\end{aligned}
$$

where

$$K_1 = \bar{x}_0^2 e^{\nu T} + \left(\frac{k\bar{x}_0}{\beta - \nu + z} - \frac{kc(\beta - r)y_0}{(\beta - z)(\beta - \nu + z)} \right)(e^{(\beta+z)T} - e^{\nu T}) \\ + \frac{kc(\beta - r)y_0}{(\beta - z)(2\beta - \nu)}(e^{2\beta T} - e^{\nu T}),$$

$$K_2 = \left(\frac{k\theta^2}{z(\beta - \nu)} - \frac{gc(\beta - r)y_0}{(\beta - z)(\beta - \nu)} \right)(e^{\beta T} - e^{\nu T}) \\ + \left(\frac{gc(\beta - r)y_0}{(\beta - z)(z - \nu)} - \frac{g\bar{x}_0}{z - \nu} \right)(e^{zT} - e^{\nu T}) - \frac{k\theta^2}{z(\beta - \nu + z)}(e^{(\beta+z)T} - e^{\nu T})$$

and

$$K_3 = \left(\frac{\theta^2}{z\nu} - \frac{\theta^2}{\gamma^2\nu} \right)(1 - e^{\nu T}) + \frac{g\theta^2}{z(z - \nu)}(e^{zT} - e^{\nu T}).$$

The variance of the PPM's portfolio under PUF is obtained as

$$\begin{aligned} &Var(E(\bar{X}_P^*(T))) = E(\bar{X}_P^{*2}(T)) - (E(\bar{X}_P^*(T)))^2 \\ &= q_1(T) + q_2(T)E(X^*(T)) + q_3(T)(E(X^*(T)))^2 \end{aligned} \tag{6.17}$$

Setting

$$q_1(T) = K_1 + (\frac{1}{2\psi} - cy_0e^{\beta t})K_2 + (\frac{1}{2\psi} - cy_0e^{\beta T})^2 K_3$$

$$q_2(T) = (K_2 + 2(\frac{1}{2\psi} - cy_0e^{\beta T})K_3),$$

$$q_3(T) = K_3 - 1.$$

(6.17) becomes

$$Var(\bar{X}_P^*(T)) = q_1(T) + q_2(T)E(\bar{X}_P^*(T)) + q_3(T)(E(\bar{X}_P^*(T)))^2. \tag{6.18}$$

The standard deviation now becomes

$$\sigma(\bar{X}_P^*(T)) = \sqrt{q_1(T) + q_2E(\bar{X}_P^*(T)) + q_3(T)(E(\bar{X}_P^*(T)))^2}. \tag{6.19}$$

If $Var(\bar{X}_P^*(T)) = 0$, it implies that $q_3(T)(E(\bar{X}_P^*(T)))^2 + q_2(T)E(\bar{X}_P^*(T)) + q_1(T) = 0$. It then follows that

$$E(\bar{X}_P^*(T)) = \frac{-q_2(T) \pm \sqrt{q_2(T)^2 - 4q_1(T)q_3(T)}}{2q_3(T)}. \tag{6.20}$$

Observe from (6.20) that if $q_2(T)^2 \geq 4q_1(T)q_3(T)$, the expected terminal wealth will have real values, and real and imaginary values when $q_2(T)^2 < 4q_1(T)q_3(T)$. (6.20) gives the expected terminal wealth of the PPM when the portfolio is free from risky under PUF.

Figure 9 shows the efficient frontier of the PPM under PUF. It is observe that PPM will stand to have 1 to 10 million and stand the risk of losing about 1 to 8 million.

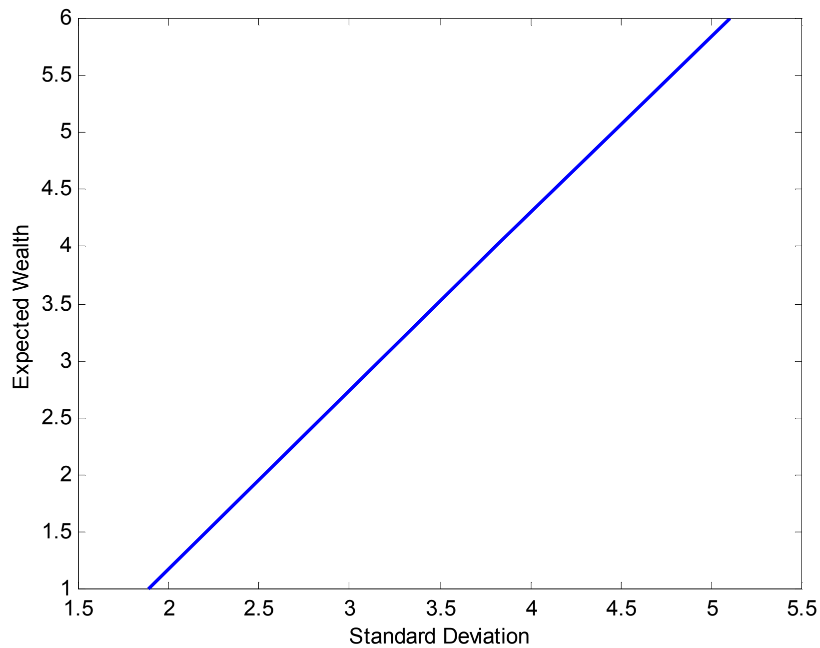

FIGURE 9. Efficient Frontier under PUF. This is obtained by setting $\mu = 0.09$, $\sigma = 0.3$, $\beta = 0.0292$, $c = 0.15$, $y_0 = 0.9$, $T = 20$, $r = 0.04$, $\gamma = 2$, $\psi = 1$, $x_0 = 0.865$.

7. The Special Case, $c = 0$ (or $y_0 = 0$)

By set $c = 0$, we obtained the usual portfolio selection problem. It is obvious from the previous analysis, that inequalities still hold for $c = 0$ provided that the initial wealth is greater than zero. We now summarize the expected terminal wealth of the investor, taking $c = 0$ or $y_0 = 0$ in (4.30), (5.12) and (6.10).

Corollary 7.1. *Assume that an investor wants to invest a wealth of $x_0 > 0$ for the time horizon $T > 0$ in a financial market as in section (2.1) and wealth equation (2.5). Assume that the investor maximizes the expected utility of final wealth at time T under QUF, EUF and PUF. Then,*

(i) $E(\bar{X}^*(T)) = x_0 e^{rT} + \dfrac{e^{\theta^2 T} - 1}{2\psi}$.

(ii) $E(\bar{X}_e^*(T)) = x_0 e^{rT} + \dfrac{\theta^2 T}{\alpha}$;

(iii) $E(\bar{X}_P^*(T)) = \dfrac{r\gamma + \theta^2}{r\gamma + \theta^2 - \gamma\theta^2(1 - e^{(r+\frac{\theta^2}{\gamma})T})}[x_0 e^{(r+\frac{\theta^2}{\gamma})T} + \dfrac{\gamma\theta^2(1 - e^{(r+\frac{\theta^2}{\gamma})T})}{2\psi(r\gamma + \theta^2)}]$.

From Corollary 7.1, if we allow ψ to be considerably large (i.e., $\psi \to \infty$), then

$$E(\bar{X}_P^*(T)) = \frac{(r\gamma + \theta^2)x_0 e^{(r+\frac{\theta^2}{\gamma})T}}{r\gamma + \theta^2 - \gamma\theta^2(1 - e^{(r+\frac{\theta^2}{\gamma})T})}; \tag{7.1}$$

$$E(\bar{X}^*(T)) = x_0 e^{rT}; \tag{7.2}$$

$$E(\bar{X}_e^*(T)) = x_0 e^{rT} + \frac{\theta^2 T}{\alpha}. \tag{7.3}$$

Observe from (5.12) that $E(\bar{X}_e^*(T))$ does not depend on ψ, so the increase or decrease in ψ do not affect it. Also, observe from (7.2) and (7.3) that $Var(\bar{X}_e^*(T)) = E(\bar{X}_e^*(T)) - E(\bar{X}^*(T)) = \frac{\theta^2 T}{\alpha}$. This implies that if α is considerably small, the difference between $E(\bar{X}_e^*(T))$ and $E(\bar{X}^*(T))$ will be very large and vice versa. Observe from (7.1) and (7.2) that $\frac{E(\bar{X}_P^*(T))}{E(\bar{X}^*(T))} = \frac{(r\gamma+\theta^2)e^{\frac{\theta^2}{\gamma}T}}{r\gamma+\theta^2-\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}$. We then have the following propositions.

Proposition 7.2. *Suppose that Corollary 7.1 holds,* $\lambda = \frac{(r\gamma+\theta^2)}{r\gamma+\theta^2-\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}$, *then*
(i) $E(\bar{X}^*(T)) = E(\bar{X}_e^*(T))$ *if and only if* $\psi^* = \frac{\alpha}{2\theta^2 T}(e^{\theta^2 T}-1)$;
(ii) $E(\bar{X}^*(T)) > E(\bar{X}_e^*(T))$ *if and only if* $\psi^* < \frac{\alpha}{2\theta^2 T}(e^{\theta^2 T}-1)$;
(iii) $E(\bar{X}^*(T)) < E(\bar{X}_e^*(T))$ *if and only if* $\psi^* > \frac{\alpha}{2\theta^2 T}(e^{\theta^2 T}-1)$;
(iv) $E(\bar{X}^*(T)) = E(\bar{X}_P^*(T))$ *if and only if* $\psi^* = \frac{\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})-(r\gamma+\theta^2)(e^{\theta^2 T}-1)}{2x_0e^{rT}(r\gamma+\theta^2)(1-\lambda e^{\frac{\theta^2 T}{\gamma}})}$;
(v) $E(\bar{X}^*(T)) > E(\bar{X}_P^*(T))$ *if and only if* $\psi^* > \frac{\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})-(r\gamma+\theta^2)(e^{\theta^2 T}-1)}{2x_0e^{rT}(r\gamma+\theta^2)(1-\lambda e^{\frac{\theta^2 T}{\gamma}})}$;
(vi) $E(\bar{X}^*(T)) < E(\bar{X}_P^*(T))$ *if and only if* $\psi^* < \frac{\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})-(r\gamma+\theta^2)(e^{\theta^2 T}-1)}{2x_0e^{rT}(r\gamma+\theta^2)(1-\lambda e^{\frac{\theta^2 T}{\gamma}})}$.

7.1. **Linear Relationship Between the Utility Functions.** In this subsection, we presents the linear relationship between PUF and QUF, EUF and QUF, and PUF and EUF. The following propositions establish this facts.

Proposition 7.3. *Suppose that Corollary 7.1 holds, then*
(i) $E(\bar{X}_P^*(T)) = \lambda\left(\frac{\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}{2\psi(r\gamma+\theta^2)} - \frac{e^{\frac{\theta^2 T}{\gamma}}(e^{\theta^2 T}-1)}{2\psi}\right) + \lambda e^{\frac{\theta^2 T}{\gamma}} E(\bar{X}^*(T))$;
(ii) $E(\bar{X}_e^*(T)) = \frac{2\psi\theta^2 T - \alpha(e^{\theta^2 T}-1)}{2\psi\alpha} + E(\bar{X}^*(T))$;
(iii) $E(\bar{X}_P^*(T)) = \lambda\theta^2\left(\frac{\gamma(1-e^{(r+\frac{\theta^2}{\gamma})T})}{2\psi(r\gamma+\theta^2)} - \frac{Te^{\frac{\theta^2 T}{\gamma}}}{\alpha}\right) + \lambda e^{\frac{\theta^2 T}{\gamma}} E\bar{X}_e^*(T)$.

Proposition 7.4. *Suppose that (7.1), (7.2) and (7.3) hold, then*
(i) $Var(\bar{X}_e^*(T)) = E(\bar{X}_e^*(T)) - E(\bar{X}^*(T))$;
(ii) $E(\bar{X}_P^*(T)) = \lambda e^{\frac{\theta^2 T}{\gamma}} E(\bar{X}^*(T))$;
(iii) $E(\bar{X}_e^*(T)) = \frac{\theta^2 T}{\alpha} + E(\bar{X}^*(T))$;
(iv) $E(\bar{X}_P^*(T)) = -\frac{\lambda\theta^2 T e^{\frac{\theta^2 T}{\gamma}}}{\alpha} + \lambda e^{\frac{\theta^2 T}{\gamma}} E\bar{X}_e^*(T)$.

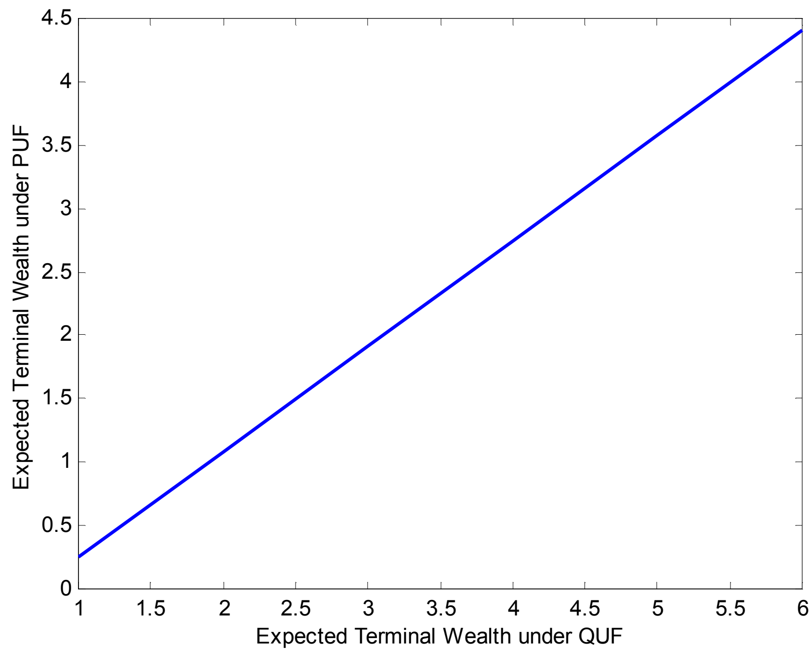

FIGURE 10. PUF versus QUF on PPM's Final Wealth.

Proposition 7.3(i) shows that a linear relationship exists between the expected final funds under PUF and QUP with intercept $\lambda(\frac{\gamma\theta^2(1-e^{(r+\frac{\theta^2}{\gamma})T})}{2\psi(r\gamma+\theta^2)}-\frac{e^{\frac{\theta^2T}{\gamma}}(e^{\theta^2T}-1)}{2\psi})$ and gradient $\lambda e^{\frac{\theta^2T}{\gamma}}$. Similarly, Proposition 7.3(ii) shows that a linear relationship exists between the expected final funds under EUF and QUF with intercept $\frac{2\psi\theta^2T-\alpha(e^{\theta^2T}-1)}{2\psi\alpha}$ and gradient one. Also, Proposition 7.3(iii) shows that there is a linear relationship between the expected final fund under PUF and EUF with intercept $\lambda\theta^2(\frac{\gamma(1-e^{(r+\frac{\theta^2}{\gamma})T})}{2\psi(r\gamma+\theta^2)}-\frac{Te^{\frac{\theta^2T}{\gamma}}}{\alpha})$ and gradient $\lambda e^{\frac{\theta^2T}{\gamma}}$.

The following figures were obtain by setting $\mu=0.09$, $\sigma=0.3$, $\beta=0.0292$, $c=0.15$, $y_0=0.9$, $T=20$, $r=0.04$, $\gamma=0.9$, $x_0=0.865$, $\alpha=0.5$, $\psi=1$.

Figure 10 shows the linear relationship between final wealth for PUF and QUF. It is observe that when the final wealth under PUF increases by about 83%, it will lead to about 100% increase in final wealth under QUF. It implies that the ratio of increase between PUF and QUF is 8.3 : 10.

Figure 11 shows the linear relationship between final wealth for EUF and QUF. It is observe that at about 100% increase in the final wealth from QUF will lead to about 180% increase in the final wealth of EUF, that is a ratio of 10 : 18.

Figure 12 shows the linear relationship between final wealth for PUF and EUF. It was found that at 100% increase in the final wealth under EUF leads to about 83% increase in the final wealth under PUF.

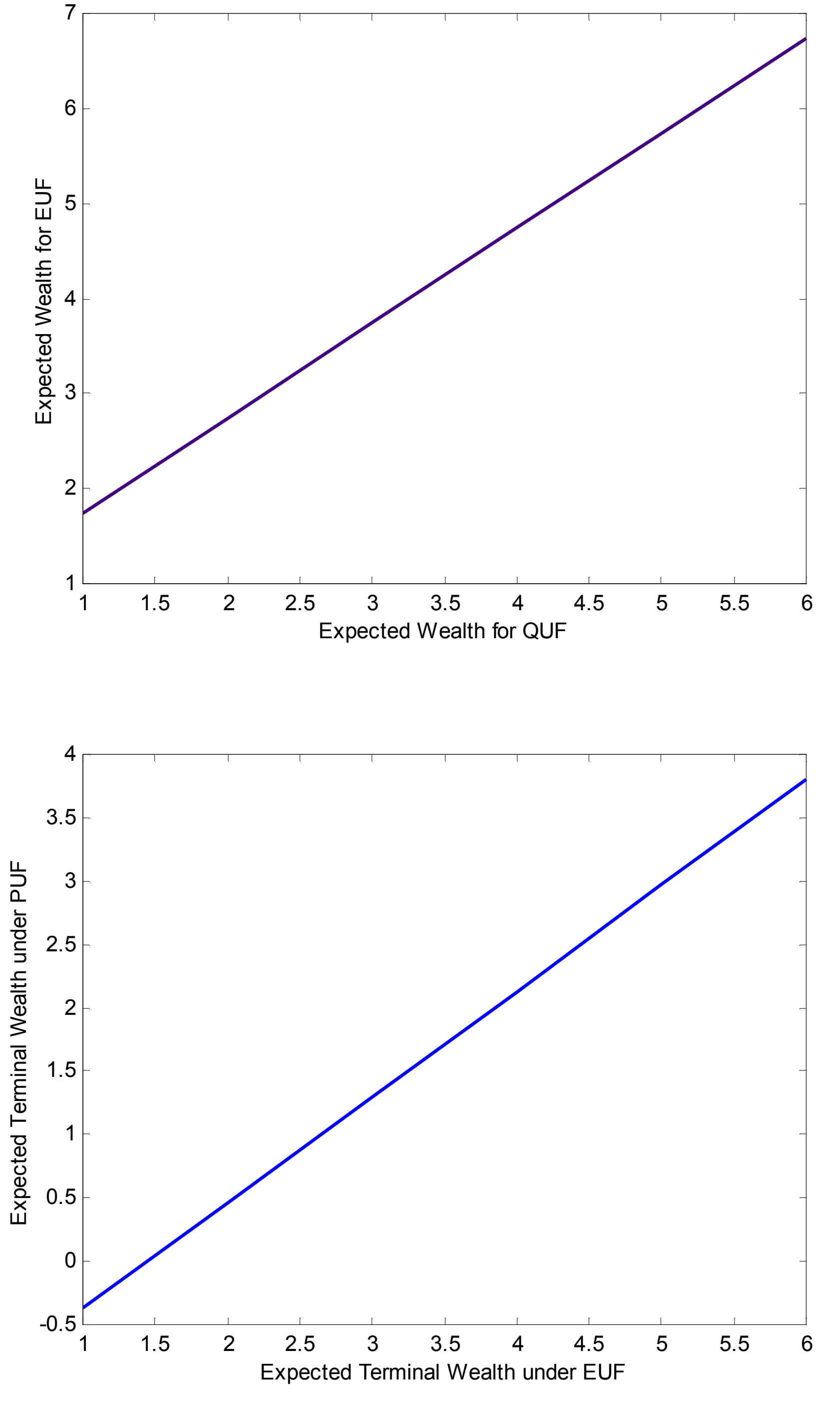

Figure 12. PUF versus EUF on PPM's Final Wealth.

8. Conclusion

In this paper, we examined a mean-variance portfolio selection problem with time-dependent salary in the accumulation phase of a defined contribution (DC) pension scheme. The optimal portfolio processes and expected wealth for the PPM were established. The efficient frontier of a PPM portfolio in mean-standard deviation under Quadratic Utility Function (QUF), Power Utility Function (PUF) and Exponential Utility Function (EUF) were established. The trade off between the expected final wealth for the PPM under PUF and QUF, EUF and QUF and PUF and EUF were established.

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